Solution Manual
for
Robust Control: Theory and Applications

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The following conventions are used throughout the solution manual:

CLS: closed loop system

iff: if and only if

Chapter 2

Problem 2.1
The equation follows from

$$\det \begin{bmatrix} I_n & B \\ -C & I_m \end{bmatrix} = \det(I_m)\det(I_n + B \cdot I_m^{-1} \cdot C) = \det(I_n)\det(I_m + C \cdot I_n^{-1} \cdot B)$$

Further, since cb is a scalar we have $\det(1 + cb) = 1 + cb$.

Problem 2.2
When $x, y \in \mathbb{R}^n$ and $a, b \in \mathbb{R}$, we have $ax, by \in \mathbb{R}^n$ due to property (b). Then, by property (a) there holds $ax + by \in \mathbb{R}^n$. Conversely, when $ax + by \in \mathbb{R}^n$ for any $x, y \in \mathbb{R}^n$ and $a, b \in \mathbb{R}$, (a) follows by setting $a = b = 1$ and $b = 0$ leads to (b).

Problem 2.3

$$\begin{aligned} \|x\|_1 &= |1| + |2| + |3| = 6 \\ \|x\|_2 &= \sqrt{1^2 + 2^2 + 3^2} = 14 \\ \|x\|_\infty &= \max\{|1|,\ |2|,\ |3|\} = 3 \end{aligned}$$

Problem 2.4

$$\begin{aligned} \|u\|_2 &= \sqrt{0^2 + 1^2} = 1 \\ \|v\|_2 &= \sqrt{1^2 + 1^2} = \sqrt{2} \\ \langle u, v \rangle &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \\ \cos\theta &= \frac{\langle u, v \rangle}{\|u\|_2 \|v\|_2} = \frac{1}{\sqrt{2}} \ \Rightarrow\ \theta = \frac{\pi}{4} \end{aligned}$$

Problem 2.5
It is easy to see that conditions (1) and (2) are satisfied. As for condition (3), it follows from $0 = \|x\|_2^2 = x_1^2 + x_2^2 + x_3^2 = 0 \Leftrightarrow x_1 = x_2 = x_3 = 0$. Further, Cauchy-Schwarz inequality $x_1x_2 + y_1y_2 + z_1z_2 \leq \sqrt{x_1^2 + y_1^2 + z_1^2}\sqrt{x_2^2 + y_2^2 + z_2^2}$ gives the triangle inequality condition (4).

Problem 2.6

By definition, when $y_1, y_2 \in \mathrm{Im}A$ there exist $x_1, x_2 \in \mathbb{F}^n$ such that $y_1 = Ax_1, y_2 = Ax_2$. Since $\alpha x_1 + \beta x_2 \in \mathbb{F}^n$ for any $\alpha, \beta \in \mathbb{F}$, $\alpha y_1 + \beta y_2 = \alpha Ax_1 + \beta Ax_2 = A(\alpha x_1 + \beta x_2) \in \mathrm{Im}A$ is true. This means that $\mathrm{Im}A \subset \mathbb{F}^m$ is a subspace.
Similarly, $x_1, x_2 \in \mathrm{Ker}A \Rightarrow Ax_1 = Ax_2 = 0 \Rightarrow A(\alpha x_1 + \beta x_2) = \alpha Ax_1 + \beta Ax_2 = 0 \Rightarrow \alpha x_1 + \beta x_2 \in \mathrm{Ker}A$. That is, $\mathrm{Ker}A \subset \mathbb{F}^n$ is also a subspace.

Problem 2.7

The two row vectors of matrix A are obviously linearly independent. So $\mathrm{rank}A = 2$. For any nonzero vector $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$,

$$Ax = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x_3.$$

So, the bases of $\mathrm{Im}A$ are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and that of $\mathrm{Ker}A$ is $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Problem 2.8

First, $b \in \mathrm{Im}A$ since

$$b = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}.$$

Further, the matrix A has full column rank so that the solution is unique. In fact, it is $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. However, for $b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathrm{rank}[A\ b] = 3 > \mathrm{rank}(A) = 2$. So no solution exists in this case.

Problem 2.9

$x_4 = 1$ is obvious. Solving for x_2 first, then x_1, we get $x = \begin{bmatrix} x_3 - 1 \\ -2x_3 \\ x_3 \\ 1 \end{bmatrix}$ in which x_3 is arbitrary. That is, the solution has one free parameter.

Problem 2.10

Writing the equation in a vector form w.r.t. the unknown vector $\begin{bmatrix} u[n-1] \\ \vdots \\ u[0] \end{bmatrix}$, it is straightforward to see that rank $[b \ \ Ab \ \ \cdots \ \ A^{n-1}b] = n$ is the necessary and sufficient condition because $x[0], \ x[n]$ are arbitrary.

Problem 2.11

(a) $\frac{d}{dx}(Ax-b)^T(Ax-b) = 2x^TA^TA - 2b^TA = 0 \ \Rightarrow \ x = (A^TA)^{-1}A^Tb$.

(b) Set a Lagrange function as $J = x^Tx + \lambda^T(Ax-b)$ in which λ is the multiplier. The solution is obtained from solving the simultaneous equations:

$$\frac{\partial J}{\partial x} = 2x^T + \lambda^T A = 0, \quad \frac{\partial J}{\partial \lambda} = (Ax-b)^T = 0.$$

Solving for x from the first equation, we get $x = -A^T\lambda/2$. Its substitution into the second equation leads to $AA^T\lambda = -2b \Rightarrow \lambda = -2(AA^T)^\dagger b$. So finally $x = A^T(AA^T)^\dagger$.

Problem 2.12

For matrix B, the eigenvalues $\lambda_1 = \sqrt{3}$ and $\lambda_2 = -\sqrt{3}$ are obtained from solving the characteristic equation

$$|\lambda I - B| = \lambda^2 - 3 = 0.$$

Meanwhile, the corresponding eigenvectors $u_1 = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$ and $u_2 = \begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix}$ come from the definition of eigenvector:

$$(\lambda I - B)u = 0.$$

For matrix A, its eigenvalues are $\lambda_1 = -3, \ \lambda_2 = 1$ and the eigenvectors are $v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Problem 2.13

The eigenvalues are $\lambda_1 = \lambda_2 = 1, \lambda_3 = 0$ and there are two eigenvectors $u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, u_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and a generalized eigenvector $u_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

Setting a transformation matrix as $T = [u_1 \ u_2 \ u_3]$, we have

$$T^{-1}AT = \begin{bmatrix} J_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad J_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Calculation based on $T^{-1}A^kT = (T^{-1}AT)^k = \mathrm{diag}(J_1^k, 0)$ yields

$$J_1^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \Rightarrow A^k = T \cdot \mathrm{diag}(J_1^k,\ 0) \cdot T^{-1} = \begin{bmatrix} 1 & 1 & k-1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Problem 2.14
A function analytic in a domain containing the origin can be expanded as

$$f(\lambda) = f(0) + f'(0)\lambda + \frac{1}{2!}f^{(2)}(0)\lambda^2 + \cdots + \frac{1}{n!}f^{(n)}(0)\lambda^n + \cdots.$$

So,

$$\begin{aligned} f(A)x &= \left[f(0)I + f'(0)A + \frac{1}{2!}f^{(2)}(0)A^2 + \cdots + \frac{1}{n!}f^{(n)}(0)A^n + \cdots\right]x \\ &= f(0)x + f'(0)\lambda x + \frac{1}{2!}f^{(2)}(0)\lambda Ax + \cdots + \frac{1}{n!}f^{(n)}(0)\lambda A^{n-1}x + \cdots \\ &= f(0)x + f'(0)\lambda x + \frac{1}{2!}f^{(2)}(0)\lambda^2 x + \cdots + \frac{1}{n!}f^{(n)}(0)\lambda^2 A^{n-2}x + \cdots \\ &\vdots \\ &= f(0)x + f'(0)\lambda x + \frac{1}{2!}f^{(2)}(0)\lambda^2 x + \cdots + \frac{1}{n!}f^{(n)}(0)\lambda^n x + \cdots \\ &= f(\lambda)x. \end{aligned}$$

Problem 2.15
(a) Suppose the opposite, i.e., $q_1, \ldots, q_n$ are linearly dependent. Then, there exist $\alpha_1,\ \ldots,\ \alpha_n$ and at least one of them is nonzero such that $\alpha_1 q_1 + \cdots + \alpha_n q_n = 0$ holds. Pre-multiplying A to this equation repeatedly, we have $\alpha_1\lambda_1^k q_1 + \cdots + \alpha_n\lambda_n^k q_n = 0$ ($k = 0,\ 1,\ 2,\ \ldots$). So,

$$\begin{bmatrix} \alpha_1 q_1 & \cdots & \alpha_n q_n \end{bmatrix} \begin{bmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \lambda_n & \cdots & \lambda_n^{n-1} \end{bmatrix} = 0.$$

It is well known that the second matrix is nonsingular when all λ_i are distinct. Then, all $\alpha_i q_i$ are zero vectors and at least one eigenvector q_k is zero. This contradicts the definition of eigenvector. So, $q_1, \ldots, q_n$ must be linearly independent and as the result Q is nonsingular.
(b) $AQ = Q\mathrm{diag}\cdot(\lambda_1,\ \cdots,\ \lambda_n) \Rightarrow PA = \mathrm{diag}(\lambda_1,\ \cdots,\ \lambda_n)P \Rightarrow p_iA = \lambda_i p_i$.

Problem 2.16
Owing to $Ax = \lambda x \Rightarrow x^*A^* = x^*A = \overline{\lambda}x^*$, there holds $\lambda x^*x = x^*Ax = (x^*Ax)^* = \overline{\lambda}x^*x \Rightarrow \overline{\lambda} = \lambda$. So $\lambda \in \mathbb{R}$. Further, $\lambda x^*x = x^*Ax \geq 0$ when $A \geq 0$, so $\lambda \geq 0$.

Problem 2.17
Let $S = \text{Im } T_1$ and T_1 have full column rank. According to Theorem 2.7(1) there is a matrix A_{11} satisfying $AT_1 = A_{11}T_1$. Suppose that λ is an eigenvalue of A_{11}, its eigenvector being $u \neq 0$. Then, $A_{11}u = \lambda u \Rightarrow AT_1u = \lambda T_1u$ holds. Since T_1 has full column rank, $x = T_1u \neq 0 \in S$ becomes an eigenvector of A.

Problem 2.18
$A_1 > 0$ follows immediately from a direct application of Schur's lemma, i.e.

$$1 > 0, \quad 2 - 1 \times (1)^{-1} \times 1 = 1 > 0.$$

In proving $A_2 > 0$, we may reduce the 3-dimensional problem into an equivalent 2-dimensional one:

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \times (2)^{-1} \times \begin{bmatrix} 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

by using Schur's lemma. Then, application of Schur's lemma again on this new problem leads to the conclusion.

Problem 2.19
For matrix B, the singular values satisfying

$$|\sigma^2 I - B^T B| = (\sigma^2 - 1)(\sigma^2 - 9) = 0.$$

are $\sigma_1 = 3$ and $\sigma_2 = 1$, their corresponding singular vectors satisfy

$$(\sigma^2 I - B^T B)v = 0$$

and are obtained as $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
As for matrix A, its singular values are $\sigma_1 = \sqrt{7 + 2\sqrt{10}}$, $\sigma_2 = \sqrt{7 - 2\sqrt{10}}$. The singular vectors are omitted as they are messy.

Problem 2.20
According to Problem 2.16, the eigenvalue λ_i in $Ax_i = \lambda_i x_i$ is a real number. So, from $A^* A x_i = \lambda A^* x_i = \lambda_i A x_i = \lambda_i^2 x_i$ we see that the singular value σ_i satisfies the relation

$$\sigma_i = |\lambda_i|.$$

Problem 2.21

$$\begin{aligned} &\|A\|_1 = 3, \quad \|A\|_2 = \sigma_1 = \sqrt{7 + 2\sqrt{10}}, \quad \|A\|_\infty = 5 \\ &\|B\|_1 = 3, \quad \|B\|_2 = \sigma_1 = 3, \quad \|B\|_\infty = 3 \end{aligned}$$

Problem 2.22
On $\|A\|_2$:

$$\begin{aligned} \|Au\|_2^2 = u^* A^* A u \le \lambda_{\max}(A^* A)\, \|u\|_2^2 \\ \Rightarrow \frac{\|Au\|_2}{\|u\|_2} \le \sqrt{\lambda_{\max}(A^* A)}. \end{aligned}$$

Since the right hand side is independent of the arbitrary vector u, this inequality holds even if the left hand side takes its supremum. That is, $\|A\|_2 \le \sqrt{\lambda_{\max}(A^* A)}$. To show the converse, we focus on the eigenvector v of the largest singular value. In this case,

$$\|Av\|_2^2 = \lambda_{\max}(A^* A)\, \|v\|_2^2 \Rightarrow \sup \frac{\|Au\|_2}{\|u\|_2} \ge \frac{\|Av\|_2}{\|v\|_2} = \sqrt{\lambda_{\max}(A^* A)}.$$

Therefore, the equality must be true.
On $\|A\|_1$:
Let i^* be the row number at which $\sum_j |a_{i^* j}| = \max_i \sum_j |a_{ij}|$ and u_* be a special vector whose jth element is $\mathrm{sgn}(a_{i^* j})$ ($\mathrm{sgn}(a)$ is the signum function defined by $\mathrm{sgn}(a) = 1\ \forall a > 0$, $\mathrm{sgn}(0) = 0$ and $\mathrm{sgn}(a) = -1\ \forall a < 0$). Then, the conclusion follows from

$$\|Au_*\|_\infty = \sum_j |a_{i^* j}|\, \|u_*\|_\infty = \max_i \sum_j |a_{ij}|\, \|u_*\|_\infty$$

as well as

$$\begin{aligned} \|Au\|_\infty &= \max_i |\sum_j a_{ij} u_j| \le \max_i \sum_j |a_{ij}||u_j| \le \max_i \sum_j |a_{ij}| \ \max_j |u_j| \\ &= \max_i \sum_j |a_{ij}|\, \|u\|_\infty . \end{aligned}$$

Problem 2.23
Omitted.

Problem 2.24
It is derived by differentiating the two sides of $A^{-1}(t)A(t) = I$:

$$\frac{d}{dt}(A^{-1}(t)) \cdot A(t) + A^{-1}(t) \cdot \frac{d}{dt}(A(t)).$$

Problem 2.25
The results are derived by expanding each scalar function first, then calculate in accordance with the definition.
As an example, we look at

$$\begin{aligned} x^T Ax =& a_{11}x_1^2 + \cdots + a_{1i}x_1x_i + \cdots + a_{1n}x_1x_n \\ &\vdots \\ &+ a_{i1}x_ix_1 + \cdots + a_{ii}x_i^2 + \cdots + a_{in}x_ix_n \\ &\vdots \\ &+ a_{n1}x_nx_1 + \cdots + a_{ni}x_ix_n + \cdots + a_{nn}x_n^2. \end{aligned}$$

It is easy to see that

$$\begin{aligned} \frac{\partial(x^T Ax)}{\partial x_i} =& a_{1i}x_1 + \cdots + a_{(i-1)i}x_{i-1} + a_{(i+1)i}x_{i+1} + \cdots + a_{1n}x_1x_n \\ &+ a_{i1}x_1 + \cdots + 2a_{ii}x_i + \cdots + a_{in}x_n \\ =& 2\begin{bmatrix} a_{i1} & \cdots & a_{ii} & \cdots & a_{in} \end{bmatrix} x \end{aligned}$$

in which $a_{ij} = a_{ji}$ has been used. So

$$\frac{\partial(x^T Ax)}{\partial x} = 2(Ax)^T.$$

Problem 2.26
The functions in (a) and (c) satisfy all conditions on norm except $\|u\| = 0 \Leftrightarrow u = 0$ (such a function is called a semi-norm). The function in (b) is a norm.

Problem 2.27
First,

$$G(s) = \frac{-1/3}{s+2} + \frac{4/3}{s+5} \;\Rightarrow\; g(t) = -\frac{1}{3}e^{-2t} + \frac{4}{3}e^{-5t}.$$

Then $\|G\|_2 = \|g\|_2 = \frac{1}{30}\sqrt{\frac{523}{7}}$ is calculated based on the definition of $\|g\|_2$. $\frac{d}{dt}|G(j\omega)|^2 = 0$ has solutions $\omega^2 = 0,\ 6\sqrt{2}-1,\ \infty$. But the maximum is taken at $\omega = \sqrt{6\sqrt{2}-1}$. So, $\|G\|_\infty \approx 0.151$.

Problem 2.28
First,

$$G(s) = \frac{4/9}{s+1} + \frac{5/9}{s+10} \;\Rightarrow\; g(t) = \frac{4}{9}e^{-t} + \frac{5}{9}e^{-10t}.$$

Then $\|G\|_2 = \|g\|_2 = \sqrt{7/44}$ is obtained by integration.
Second,

$$G(s)G(-s) = \frac{(s+5)(-s+5)}{(s+1)(s+10)(-s+1)(-s+10)}$$

has two poles $-1,\ -10$ in the RHP. So,

$$\begin{aligned}\|G\|_2^2 &= \left.\frac{(s+5)(-s+5)}{(s+10)(-s+1)(-s+10)}\right|_{s=-1} + \left.\frac{(s+5)(-s+5)}{(s+1)(-s+1)(-s+10)}\right|_{s=-10} \\ &= \frac{7}{44}.\end{aligned}$$

Problem 2.29
Note that $|D(j\omega)| = 1 \Rightarrow |D(j\omega)G(j\omega)| = |G(j\omega)|$, $|A(j\omega)| = 1 \Rightarrow |A(j\omega)G(j\omega)| = |G(j\omega)|$. The conclusions follow from the definition of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms directly.

Problem 2.30
Noting $(\hat{f}^*\hat{g})^* = \hat{g}^*\hat{f}$, it is trivial to show that

$$\left\langle \hat{f}, \hat{g} \right\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}^*(j\omega)\hat{g}(j\omega)d\omega$$

satisfies $\left\langle \hat{f}, a\hat{g} + b\hat{h} \right\rangle = a\left\langle \hat{f}, \hat{g} \right\rangle + b\left\langle \hat{f}, \hat{h} \right\rangle$, $\left\langle \hat{f}, \hat{g} \right\rangle = \overline{\left\langle \hat{g}, \hat{f} \right\rangle}$. Further, $\left\langle \hat{f}, \hat{f} \right\rangle = \left\|\hat{f}\right\|_2^2 \geq 0$, and $\left\langle \hat{f}, \hat{f} \right\rangle = 0$ iff $\hat{f}(\omega) \equiv 0$.

Problem 2.31
1 and 2 are trivial. We prove 3 and 4.

$$\begin{aligned}\left\langle \hat{f}, H\hat{g} \right\rangle &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}^*(j\omega)\cdot H(j\omega)\hat{g}(j\omega)d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} (H^*(j\omega)\hat{f}(j\omega))^*\hat{g}(j\omega)d\omega \\ &= \left\langle H^{\sim}\hat{f}, \hat{g} \right\rangle.\end{aligned}$$

From this property, it is immediate that $\left\|A\hat{f}\right\|_2^2 = \left\langle A\hat{f}, A\hat{f} \right\rangle = \left\langle A^{\sim}A\hat{f}, \hat{f} \right\rangle = \left\langle \hat{f}, \hat{f} \right\rangle = \left\|\hat{f}\right\|_2^2$.

Problem 2.32
$\left\langle \hat{f}, \hat{g} \right\rangle = 0$ is proved by using the residue method. Since an antistable function is strictly proper, its value at $s = re^{j\theta} \to 0$ as $r \to \infty$. So,

$$\left\langle \hat{f}, \hat{g} \right\rangle = \oint_C \hat{f}^{\sim}(s)\hat{g}(s)ds$$

holds in which C is a closed integration path consisting of the imaginary axis and a half circle in the left half plane with an infinity radius. This integral is zero because $\hat{f}^{\sim}(s)\hat{g}(s)$ does not have any poles in the left half plane. Then, it follows that

$$\left\|\hat{f}+\hat{g}\right\|^2 = \left\|\hat{f}\right\|^2 + \|\hat{g}\|^2 + 2\Re\left\langle \hat{f}, \hat{g} \right\rangle = \left\|\hat{f}\right\|^2 + \|\hat{g}\|^2.$$

Chapter 3

Problem 3.1
By definition, for any $x_1, x_2 \in C$ and real numbers α_1, α_2 with $\alpha_1 + \alpha_2 = 1$, there holds $\alpha_1 x_1 + \alpha_2 x_2 \in C$. So, $\beta_1(\alpha_1 x_1 + \alpha_2 x_2) + \beta_2 x_3 \in C$ for any $x_2 \in C$ and real numbers β_1, β_2 with $\beta_1 + \beta_2 = 1$. Setting $\theta_1 = \alpha_1\beta_1, \theta_2 = \alpha_2\beta_1, \theta_3 = \beta_2$, we have $\theta_1 + \theta_2 + \theta_3 = 1$.
Conversely, given $\theta_1, \theta_2, \theta_3$ with $\theta_1 + \theta_2 + \theta_3 = 1$, we may set $\beta_2 = \theta_3$ and $\beta_1 = \theta_2 + \theta_3$. Further, we can set

$$\begin{aligned}&\alpha_1 = \frac{\theta_1}{\beta_1},\ \alpha_2 = \frac{\theta_2}{\beta_1} \text{ if } \beta_1 \neq 0\\ &\alpha_1,\ \alpha_2 \text{ arbitrary with } \alpha_1 + \alpha_2 = 1 \text{ if } \beta_1 = 0.\end{aligned}$$

Reversing the preceding argument leads to the conclusion.

Problem 3.2
For any x_1, x_2 in the half-plane and $\lambda_1, \lambda_2 \geq 0$ with $\lambda_1 + \lambda_2 = 1$,

$$a^T(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 a^T x_1 + \lambda_2 a^T x_2 \leq \lambda_1 b + \lambda_2 b = b.$$

That is, $\lambda_1 x_1 + \lambda_2 x_2$ is also in the half-plane. Therefore, it is convex.

Problem 3.3
Make the following variable transformation:

$$x_1 = \frac{y_1}{\sqrt{\lambda_1}},\ x_2 = \frac{y_2}{\sqrt{\lambda_2}},\ x_3 = \frac{y_3}{\sqrt{\lambda_3}}.$$

Then, the ellipsoid is transformed into a unit ball:

$$x_1^2 + x_2^2 + x_3^2 \leq 1.$$

The volume of a unit ball is $\frac{4}{3}\pi$. Since $dy_1 dy_2 dy_3 = \sqrt{\lambda_1}dx_1 \cdot \sqrt{\lambda_2}dx_2 \cdot \sqrt{\lambda_3}dx_3 = \sqrt{\lambda_1\lambda_2\lambda_3}dx_1 dx_2 dx_3$ and the volumes are their respective integrals within the ellipsoid/ball, the volume of the ellipsoid becomes

$$\iiint dy_1 dy_2 dy_3 = \sqrt{\lambda_1\lambda_2\lambda_3} \iiint dx_1 dx_2 dx_3 = \frac{4}{3}\pi\sqrt{\lambda_1\lambda_2\lambda_3}.$$

Problem 3.4
Similar to Problem 3.2, for any $x_1, x_2 \in \mathcal{P}$ and $\lambda_1, \lambda_2 \geq 0$ with $\lambda_1 + \lambda_2 = 1$, all of the following hold:

$$a_1^T(\lambda_1 x_1 + \lambda_2 x_2) \leq b_1,\ a_2^T(\lambda_1 x_1 + \lambda_2 x_2) \leq b_2,\ c^T(\lambda_1 x_1 + \lambda_2 x_2) = d.$$

So, the set $\mathcal{P}$ is convex.

Problem 3.5
omitted.

Problem 3.6
For any $x_1, x_2 \in S_+$ and $\lambda_1, \lambda_2 \geq 0$ with $\lambda_1 + \lambda_2 = 1$, at least one of $\lambda_i X_i$ $(i = 1,\ 2)$ is positive definite. So, $\lambda_1 X_1 + \lambda_2 X_2 > 0$ always holds. Therefore, S_+ is a convex set. In fact, it is a convex cone.

Problem 3.7
(a) $\to$ (b): It is easy to see, by the properties $\mathrm{Tr}(A^T) = \mathrm{Tr}(A)$, $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ and $\mathrm{Tr}(A + B) = \mathrm{Tr}(A) + \mathrm{Tr}(B)$ of matrix trace, that $\mathrm{Tr}((M + M^T)X) = 2\mathrm{Tr}(M^T X)$. Further, any $X < 0$ can be expressed as $X = -YY^T$ by a nonsingular matrix Y, and $M + M^T \geq 0 \Rightarrow -Y^T(M + M^T)Y \leq 0$. So, $\mathrm{Tr}((M + M^T)X) = \mathrm{Tr}(-Y^T(M + M^T)Y) \leq 0$ which implies (b).
Conversely, the symmetric matrix $M + M^T$ can be diagonalized by a unitary matrix T $(T^T T = TT^T = I)$. If an eigenvalue λ_k of $M + M^T$ is negative, then for $X = -\mathrm{diag}(1, \cdots, \rho, \cdots, 1) < 0$ we have

$$\mathrm{Tr}[(M+M^T)X] = \mathrm{Tr}[T^T(M+M^T)T \cdot T^T X T] = -(\lambda_1 + \cdots + \lambda_k \rho + \cdots + \lambda_n).$$

So, as $\rho \to \infty$ $\mathrm{Tr}[(M + M^T)X] = 2\mathrm{Tr}[M^T X] < 0$ which contradicts the given condition. Therefore, $M + M^T \geq 0$.
Moreover, it is clear from the proof that the strict inequality case is also true which will be used in the next problem.

Problem 3.8
Let us consider the infeasibility problem: there is no $X > 0$ satisfying $XA + A^T X < 0$.
First of all. we point out that from the proof of Example 3.3 (with the domain restricted to $X > 0$), "there is no $X > 0$ satisfying $XA + A^T X < 0$" is equivalent to "there is no $W = W^T \geq 0$ satisfying $\mathrm{Tr}[(XA + A^T X)W] \geq 0$ for all $X > 0$".
Since $\mathrm{Tr}[(XA + A^T X)W] = 2\mathrm{Tr}(XAW) = \mathrm{Tr}[(AW + WA^T)X]$, the latter statement is equivalent to "there is no $W = W^T \geq 0$ satisfying $\mathrm{Tr}[(AW + WA^T)X] \geq 0$ for all $X > 0$". Then, by Problem 3.7 this is equivalent to "there is no $W = W^T \geq 0$ satisfying $AW + WA^T \geq 0$".

Problem 3.9
Obviously, $f(X)I - X \geq 0$ holds for any symmetric matrix X. Then for

$\alpha_1, \alpha_2 \geq$ with $\alpha_1 + \alpha_2 = 1$ and symmetric matrices X, Y,

$[\alpha_1 f(X) + \alpha_2 f(Y)]I - [\alpha_1 X + \alpha_2 Y] = \alpha_1[f(X)I - X] + \alpha_2[f(Y)I - Y] \geq 0$

holds. Since $f(\alpha_1 X + \alpha_2 Y) = \lambda_{\max}(\alpha_1 X + \alpha_2 Y)$ is the smallest number in all p satisfying $pI - [\alpha_1 X + \alpha_2 Y] \geq 0$, there must be

$$\alpha_1 f(X) + \alpha_2 f(Y) \geq f(\alpha_1 X + \alpha_2 Y),$$

i.e., $f(X)$ is convex.

Problem 3.10
According to the first-order condition, $f(x)$ is convex iff for any $x, y \in \mathbf{dom} f$,

$$f(y) - f(x) \geq \nabla f(x)(y - x).$$

Then, for any Δx

$$\begin{aligned}
&f(x) - f(x - \Delta x) \geq \nabla f(x - \Delta x)\Delta x \\
&f(x) - f(x + \Delta x) \geq -\nabla f(x + \Delta x)\Delta x \\
&\Rightarrow 2f(x) - f(x + \Delta x) - f(x - \Delta x) \geq [\nabla f(x - \Delta x) - \nabla f(x + \Delta x)]\Delta x
\end{aligned}$$

In the last inequality, as $\Delta x \to 0$, the left side tends to zero and the right side to $-2(\Delta x)^T \nabla^2 f(x)\Delta x$. This implies $\nabla^2 f(x) \geq 0$ because the direction of Δx may be arbitrary.

Problem 3.11
$X = \Re(X) + j\Im(X)$ is Hermitian iff $\Re(X)^T = \Re(X),\ \Im(X)^T + \Im(X) = 0$. Further, $X \geq 0$ is equivalent to

$$(u + jv)^*[\Re(X) + j\Im(X)](u + jv) \geq 0$$

for any real vectors $u,\ v$ with compatible dimension. Owing to $X^* = X$, the quadratic function on the left side takes real value and its expansion becomes

$$\begin{bmatrix} u \\ v \end{bmatrix}^T \begin{bmatrix} \Re(X) & -\Im(X) \\ \Im(X) & \Re(X) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

Finally, the equivalence is obtained due to

$$u + jv \neq 0 \Leftrightarrow \begin{bmatrix} u \\ v \end{bmatrix} \neq 0.$$

Problem 3.12
The transformation is done by setting

$$c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ z = \begin{bmatrix} \lambda \\ x \end{bmatrix}, \ F(z) = \begin{bmatrix} \lambda I - A(x) & 0 \\ 0 & B(x) \end{bmatrix}.$$

Problem 3.13
By Schur's lemma,

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0 \Leftrightarrow X > 0, \ Y - X^{-1} > 0.$$

Note that

$$0 < Y - X^{-1} = X^{-1/2} \cdot X^{-1/2}(XY - I)X^{1/2} \cdot X^{-1/2}$$

is equivalent to $X^{-1/2}(XY - I)X^{1/2} > 0$. This condition is equal to the eigenvalue condition $\lambda_i[X^{-1/2}(XY - I)X^{1/2}] = \lambda_i(XY - I) > 0$ (for all i). Finally, $\lambda_i(XY - I) = \lambda_i(XY) - 1 > 0$ for all i iff $\lambda_{\min}(XY) > 1$.

Chapter 4

Problem 4.1
Controllable and observable.

Problem 4.2
(a) It follows from the following controllability and observability matrices:

$$\mathcal{C} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \mathcal{O} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}.$$

(b) A block diagram can be drawn based on the dynamics $\dot{x}_1 = x_1 + u$, $\dot{x}_2 = x_2 + u$ and $y = x_1$. x_1 and x_2 have the same dynamics so that their difference can not be controlled. Meanwhile, x_2 does not reach the output and does not affect $\dot{x}_1$, so it cannot be observed.

Problem 4.3
Let the dimension of $G(s)$ be m. Owing to the given condition, z is a zero iff $\text{rank}(G(z)) < \text{normalrank}(G(s)) = m \;\Leftrightarrow\; |G(z)| = 0$.

Problem 4.4
$z = \frac{3 \pm \sqrt{13}}{2}$.

Problem 4.5
The transfer functions are calculated respectively as

$$P_u(s) = \frac{\frac{1}{J_M J_L}}{s(s^2 + \frac{k}{J_L} + \frac{k}{J_M})}, \quad P_d(s) = \frac{\frac{1}{J_L}(s^2 + \frac{k}{J_M})}{s(s^2 + \frac{k}{J_L} + \frac{k}{J_M})}.$$

$P_d(s)$ has two imaginary zeros while $P_u(s)$ has none. The two poles on the imaginary axis move toward the zeros of $P_d(s)$ as the inertial ratio J_L/J_M increases. But contrary to the case of motor speed measurement, the inertia ratio only has a limited influence on the difficulty of control because $P_u(s)$ does not have any zero.

Problem 4.6
$\text{rank}(XY) = \text{rank}(X)$ holds for nonsingular Y (as they have the same number of independent row vectors). The result follows from

$$\begin{bmatrix} A + BF - sI & B \\ C + DF & D \end{bmatrix} = \begin{bmatrix} A - sI & B \\ C & D \end{bmatrix} \begin{bmatrix} I & 0 \\ F & I \end{bmatrix}.$$

Problem 4.7
It is easily proved by switching the roles of input and output, i.e., set y as the input and u as the output.

Problem 4.8
Let the states of G_1 and G_2 be denoted as x_1 and x_2 respectively. Focusing on their input/output relations and eliminate the connecting signals, the formulae are obtained which correspond to two different alignment of states: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ or $\begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$.

Problem 4.9
(a) Sufficiency: First, $\|y(t)\|_2 \le \int_0^t \|g(\tau)\|_2 \|u(t-\tau)\|_2 \, d\tau \le c \int_0^t \|g(\tau)\|_2 \, d\tau$ since $\|u(t)\|_2$ is bounded at any t. Then the sufficiency is obvious.
(b) Necessity: For the given u, we have $y(t) = \int_0^t \|g(\tau)\|_2 \, u_1(\tau) d\tau$ and $u_1^T(\tau)u_1(\tau) = 1$. Then, we have

$$\begin{aligned} y^T(t)y(t) &= \int_0^t \|g(\tau)\|_2 \, u_1^T(\tau) d\tau \cdot \int_0^t \|g(\tau)\|_2 \, u_1(\tau) d\tau \\ &= \int_0^t \|g(\tau)\|_2 \left[\int_0^t \|g(\tau)\|_2 \, u_1^T(\tau) u_1(\tau) d\tau \right] d\tau \\ &= \int_0^t \|g(\tau)\|_2 \left[\int_0^t \|g(\tau)\|_2 \, d\tau \right] d\tau \\ &= \left[\int_0^t \|g(\tau)\|_2 \, d\tau \right]^2 . \end{aligned}$$

The necessity is a direct consequence of the stability requirement $y^T(\infty)y(\infty) < \infty$.

Problem 4.10
$\Re(\lambda_i(A)) < -\sigma \;\Leftrightarrow\; \Re(\lambda_i(A + \sigma I)) < 0 \;\Leftrightarrow\; (A + \sigma I)$ is stable. This is equivalent to the existence of a matrix $P > 0$ satisfying

$$0 > (A + \sigma I)^T P + P(A + \sigma I) = A^T P + PA + 2\sigma P.$$

Problem 4.11
The CLS is $\dot{x} = (A + BF)x$. $\Re(\lambda_i(A + BF)) < -\sigma \;\Leftrightarrow\; \exists P > 0$ such that

$$PA + A^T P + 2\sigma P + PBF + F^T B^T P < 0.$$

Substitution of $(B^TP)_\perp = P^{-1}B_\perp^T$ yields the solvability condition

$$(B_\perp^T)^T(AQ + QA^T + 2\sigma Q)B_\perp^T < 0,\ Q = P^{-1} > 0.$$

As for the variable change method, we multiply the inequality $PA + A^TP + 2\sigma P + PBF + F^TB^TP < 0$ by $Q = P^{-1}$ from left and right and get

$$AQ + QA^T + 2\sigma Q + BFQ + QF^TB^T < 0.$$

Then, the result becomes

$$AQ + QA^T + 2\sigma Q + BM + M^TB^T < 0, \quad M = FQ.$$

Problem 4.12
Using $y^Ty = x^TC^TCx = -x^T(PA + A^TP)x = -\frac{d}{dt}(x^TPx)$ and $x(\infty) = 0$, we have

$$\begin{aligned}\int_0^\infty y^T(t)y(t)dt &= -\int_0^\infty d(x^TPx)\\ &= x^T(0)Px(0) - x^T(\infty)Px(\infty)\\ &= x^T(0)Px(0).\end{aligned}$$

Problem 4.13
L_c is the solution of Lyapunov equation

$$AL_c + L_cA^T + BB^T = 0.$$

This equation can be transformed to

$$(T^{-1}AT)(T^{-1}L_cT^{-T}) + (T^{-1}L_cT^{-T})(T^{-1}AT)^T + T^{-1}BB^TT^{-T} = 0.$$

So, the controllability Grammian of the new realization is $\hat{L}_c = T^{-1}L_cT^{-T}$. It also may be shown based on the integration formula $L_c = \int_0^\infty e^{At}BB^Te^{A^Tt}dt$. $\hat{L}_o = T^TL_oT$ follows similarly.

Chapter 5

Problem 5.1
(a) Via Routh-Hurwitz criterion, the stability range is obtained as $-4 < k < 0$.
(b) According to the final value theorem of Laplace transform,

$$\hat{y}(s) = \frac{P}{1+PK}\hat{d}(s) = \frac{2}{p(s)} \Rightarrow y(\infty) = \lim_{s\to 0} s\hat{y}(s) = 0.$$

(c) Similarly,

$$\hat{e}(s) = \frac{1}{1+PK}\hat{r}(s) = \frac{2(s+1)}{p(s)} \Rightarrow e(\infty) = 0 \Rightarrow \lim_{t\to\infty} y(t) = r(t) = \sin 2t.$$

The reason is that controller K has an integrator so that both reference tracking and disturbance rejection can be achieved w.r.t. step signals.

Problem 5.2
(a) The stability condition is $k > 0$.
(b) $e(\infty) = 0$
(c) $y(\infty) = \frac{1}{k} \neq 0$
Asymptotic tracking of step reference is attained because the loop gain $L = PK$ has an integrator. Meanwhile, since the controller K does not has any integrator the step disturbance cannot be rejected completely.

Problem 5.3
(a) The same as the preceding problem
(b) $e(\infty) = 0$
(c) $y(\infty) = 0$
In the present case, the controller K has an integrator and can realize both the step reference tracking and the step disturbance rejection.

Problem 5.4
(a) Stability range: $k > 0$.
(b) As $e(\infty) = -\frac{1}{k}$, we have $|e(\infty)| = \frac{1}{k} < 0.05 \Rightarrow k > 20$.

Problem 5.5

1. Select a sufficiently rich set of frequency $\{\omega_1, \omega_2, \ldots, \omega_N\}$.

2. Apply sinusoidal input $u_i(t) = \cos \omega_i t$ on the system, and measure the steady-state output $y_i(t) = A_i \cos(\omega_i t + \phi_i)$. Then, a pair of gain and phase angle data is obtained:

$$|G(j\omega_i)| = A_i, \quad \angle G(j\omega_i) = \phi_i$$

3. Repeating Step 2 until full sets of gain and phase are obtained.

4. Determine a rational function $G(s)$ so that its frequency response fits approximately the measured data $\{|G(j\omega_i)|,\ \angle G(j\omega_i)\}$ at all frequencies.

Problem 5.6
Consult any standard textbook on the classical control.

Problem 5.7
The tracking error w.r.t. the unit step reference is $\hat{e}(s) = \frac{1}{1+L}\frac{1}{s}$. Suppose that p is a real and positive pole of L, then we have

$$\hat{e}(p) = 0 = \int_0^\infty e(t)e^{-pt}dt.$$

$e(t) > 0$ holds for all $t \in (0,\ t_0)$ when t_0 is sufficiently small. as a result, we have $e(t)e^{-pt} > 0$ before the time instant t_0. So, there must be a time interval after t_0 in which $e(t)e^{-pt} < 0 \Rightarrow e(t) < 0$. This means that overshoot occurs.

Problem 5.8
Undershoot occurs whenever the system has real and positive zeros. Since A-type undershoot happens iff the number of such zeros is odd, so when the number is even the undershoot must be B-type.

Problem 5.9
Use MATLAB. The response speed, i.e., the rise time, is rough inversely proportional to the bandwidth.

Chapter 6

Problem 6.1
(a) As $A^n B$ is a linear combination of $B, AB, \ldots, A^{n-1}B$ due to Cayley-Hamilton theorem, $A\text{Im } \mathcal{C} \subset \text{Im } \mathcal{C}$ is obvious.
(b) Then, according to Corollary 2.1 there exists a matrix $A_1 \in \mathbb{R}^{k \times k}$ such that

$$A \begin{bmatrix} q_1 & \cdots & q_k \end{bmatrix} = \begin{bmatrix} q_1 & \cdots & q_k \end{bmatrix} A_1.$$

(c) Clearly, we can find vectors $q_{k+1}, \ldots, q_n$ such that

$$T = \begin{bmatrix} q_1 & \cdots & q_k & q_{k+1} & \cdots & q_n \end{bmatrix}$$

is nonsingular. Then, by setting

$$\begin{bmatrix} A_{12} \\ A_2 \end{bmatrix} = T^{-1} A \begin{bmatrix} q_{k+1} & \cdots & q_n \end{bmatrix},$$

we have

$$AT = T \begin{bmatrix} A_1 & A_{12} \\ 0 & A_2 \end{bmatrix}.$$

Further, since any column of matrix B belongs to $\text{Im } \mathcal{C} = \text{span}\{q_1, \cdots, q_k\}$, it becomes a linear combination of $\{q_1, \cdots, q_k\}$. So, by collecting these combination coefficients as a matrix B_1 we obtain.

$$B = T \begin{bmatrix} B_1 \\ 0 \end{bmatrix}.$$

Problem 6.2
The state feedback gain is $f = -[9 \;\; 3]$.

Problem 6.3
(a) State feedback gain: $f = -[0 \;\; 3]$
(b) The designed input is $u = fx = -3 \times [0 \;\; 1]x = -3y$, which is a static output feedback.

Problem 6.4
(a) The poles of plant are $p_1 = p_2 = 2, p_3 = -1$. Since $\text{rank}[A - p_3 I \;\; b] = 2 < 3$ the pole p_3 is not controllable so that it cannot be moved by feedback. On the other hand, the assigned CLS poles do not contain $p_3 = -1$, so such pole placement is impossible.

(b) In this case, the uncontrollable pole -1 is contained in the assigned CLS poles. So, pole placement is possible and the state feedback gain is $f = -[6 \;\; 6 \;\; 0]$.
(c) In this case, the plant is fully controllable and the CLS poles can be placed arbitrarily. In fact, $f = -\frac{1}{9}[120 \;\; 62 \;\; 1]$.

Problem 6.5
(a) Unstable since the poles are $\lambda = 5, -1$.
(b) Controllable and observable
(c) State feedback gain: $f = [11 \;\; -7]$
(d) Observer gain gain: $l = \begin{bmatrix} -11 \\ 29 \end{bmatrix}$
(e) Set $D = [0 \;\; 1] \Rightarrow S = I_2$. Then

$$T = A_{22} + lA_{12} = 2 + l \times (-1) = -3 \Rightarrow l = -5.$$

Finally, the minimal order observer and the state estimate are given by

$$\dot{\overline{z}} = -3\overline{z} + 26y + u, \quad \overline{x} = \begin{bmatrix} y \\ 5y + \overline{z} \end{bmatrix}.$$

Problem 6.6
Substitution of $u = ky = kcx$ into $\dot{x} = Ax + bu$ yields $\dot{x} = (A + kbc)x$. Since $|sI - (A + kbc)| = s^2 - ks - (k+1)$ the CLS is stable if $k < -1$.

Problem 6.7
(a) State feedback gain: $f = -[2 \; 3]$
(b) Observer gain: $l = -\begin{bmatrix} 9 \\ 20 \end{bmatrix}$

Problem 6.8
(a) Expansion of the state equation leads to ($x_{12} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$)

$$\dot{x}_{12} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} x_{12} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \; \dot{x}_3 = -x_3, \quad y = x_3.$$

y does not have any information on x_{12} so the observer cannot be designed.
(b) Observer gain: $l = -\begin{bmatrix} 34/3 \\ 34 \\ 2/3 \end{bmatrix}$

(c) One may select $D = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. The rest is omitted.

Problem 6.9
$y(t) \equiv 0 \Leftrightarrow 0 \equiv \hat{y}(s) = C(sI - A - BF)^{-1}D\hat{u}(s) \Leftrightarrow C(sI - A - BF)^{-1}D \equiv 0$. Expanding this equality, we obtain

$$C(sI - A - BF)^{-1}D = \frac{CD}{s} + \frac{C(A+BF)D}{s^2} + \frac{C(A+BF)^2D}{s^3} + \cdots \equiv 0$$
$$\Leftrightarrow C(A+BF)^kD = 0 \ (\forall k = 0, 1, 2 \ldots).$$

The conclusion is then derived from Cayley-Hamilton theorem.
In an SISO system with a relative degree r, there holds $cb = cAb = \cdots = cA^{r-2}b = 0, cA^{r-1}b \neq 0$. Hence,

$$cd = 0, 0 = c(A+bf)d = cAd, 0 = c(A+bf)^2d = cA(A+bf)d = cA^2d,$$
$$\ldots, 0 = c(A+bf)^{r-1}d = cA^{r-2}(A+bf)d = cA^{r-1}d$$

and

$$cb = 0, c(A+bf)b = cAb = 0, c(A+bf)^2b = cA(A+bf)b = cA^2b = 0,$$
$$\ldots, c(A+bf)^{r-1}b = cA^{r-2}(A+bf)b = cA^{r-1}b \neq 0$$

must be true. When it holds, $0 = c(A+bf)^rd = c(A+bf)^{r-1}Ad + c(A+bf)^{r-1}bfd = cA^{r-1}d + cA^{r-1}bfd \Rightarrow f = -(cA^{r-1}b)^{-1}cA^r$. Then, for any higher order $k > r$, $c(A+bf)^kd = cA^{r-1}(A+bf)^{k-r+1}d = (cA^r + cA^{r-1}bf)(A+bf)^{k-r}d = 0$ and the disturbance rejection condition is satisfied.

Problem 6.10
Owing to the structure of S, we have $CS^{-1} = \begin{bmatrix} I & 0 \end{bmatrix}$. Then

$$\begin{bmatrix} SAS^{-1} - \lambda I \\ CS^{-1} \end{bmatrix} = \begin{bmatrix} A_{11} - \lambda I & A_{12} \\ A_{21} & A_{22} - \lambda I \\ I & 0 \end{bmatrix}.$$

So obviously $\begin{bmatrix} A_{12} \\ A_{22} - \lambda I \end{bmatrix}$ must have full column rank for any λ. This implies the observability of $(A_{12},\ A_{22})$.

Chapter 7

Problem 7.1
$e(\infty) = 0 \Leftrightarrow Q(0) = 1/P(0) = 2$ must be satisfied. A first-order controller $K(s) = 2(s+2)/s$ is obtained by setting $Q(s) = 2$. Another solution is $Q(s) = P^{-1}(s)/(\epsilon s + 1) \Rightarrow K(s) = (s+2)/\epsilon s$.

Problem 7.2
Since $P(0)Q(0) = 1$, asymptotic tracking of step reference is guaranteed automatically. For the ramp reference,

$$e(\infty) = \lim_{s\to 0} s\hat{e}(s) = \lim_{s\to 0} \frac{(1-PQ)}{s} = \lim_{s\to 0} \frac{\epsilon}{\epsilon s + 1} == \epsilon.$$

So, $|e(\infty)| \leq 0.05 \Rightarrow 0 < \epsilon \leq 0.05$ and $K(s) = 20(s+2)/s$.

Problem 7.3

$$e(\infty) = 0 \Leftrightarrow \hat{e}(s) = (1-PQ)\frac{1}{s^2} = \frac{s^2 + (3-a)s + (2-b)}{s^2(s+1)(s+2)}$$

must be stable, which in turn requires $3 - a = 2 - b = 0$. So, $a = 3,\ b = 2$ and $K(s) = (3s+2)(s+2)/s^2$.

Problem 7.4
(a) Omitted
(b) $\hat{e}(s) = (1-PQ)/s$
(c) $e(\infty) = 0 \Leftrightarrow \hat{e}(s) = (as + b - 1)/s(as + b)$ must be stable. So $b = 1$.
(d) Since $e(t) = e^{-t/a}$ $(\forall t \geq 0)$,

$$\|e\|_2^2 = \int_0^\infty e^{-2t/a} = a/2 \leq 0.1^2 \Rightarrow 0 < a \leq 0.02.$$

(e) When $a = 0.02$, the controller is $K(s) = 10(s+5)/s$.

Problem 7.5
(a) The block diagram can be converted into a standard one w.r.t (P, K) in which $K = Q/(1 - P_0Q) = Q/(1 - PQ)$. This is the formula of stabilizing controllers for $P(s)$. So, the CLS is stable when so is $Q(s)$.
Alternative proof (based on the definition of internal stability):
Add a disturbance at the input port of the plant $P(s)$. Then,
(b) $P(j\omega)Q(j\omega) \to 1$ is necessary for ensuring $\hat{e}(j\omega) = (1 - PQ)\hat{r}(j\omega) \to 0$ over the frequency band in which $\hat{r}(j\omega)$ has a big amplitude. Since the

relative degree of P is $r > 0$, $PQ = 1/(\epsilon s+1)^r \Rightarrow Q = P^{-1}/(\epsilon s+1)^r$ is implementable. By setting ϵ small enough, the bandwidth $0 < \omega < 1/\epsilon$ widens and the tracking performance is improved.
(c) P and P_0 are in parallel connection. So, their common unstable is uncontrollable and the CLS cannot be stabilized by this IMC structure.

Problem 7.6
$H_{yw} = \frac{P}{(1+PK)}\frac{1}{s} = P(1-PQ)\frac{1}{s}$ is stable iff $Q(0) = 1/P(0) \Rightarrow b = 1$. So, $H_{yw} = a/(s+1)(as+1)$. It is low-pass, then $\|H_{yw}\|_\infty = |H_{yw}(j0)| = a < 1$. Finally, the condition is obtained as $a < 1, b = 1$.

Problem 7.7
Based on the controller obtained in Example 7.6, we have

$$1-PK = \left(1+\frac{1}{s+2}Q\right)^{-1}\frac{(s+1)^2}{s(s+2)} \Rightarrow \hat{y}(s) = \frac{P}{1-PK}\hat{d}(s) = \frac{(s+2)\left(1+\frac{1}{s+2}Q\right)}{s(s+1)^2}$$

after tedious calculation. In order to guarantee the boundedness of $\|y\|_2$, $Q(0) = -2$ is necessary. We try a choice $Q(s) = -(s+2)/(\epsilon s+1)$ and get

$$\hat{y}(s) = \frac{s+2}{(s+1)^2(s+1/\epsilon)} = \frac{a}{s+1/\epsilon} + \frac{a}{(s+1)^2} + \frac{a}{s+1}$$

where

$$a = \frac{\epsilon(2\epsilon-1)}{(\epsilon-1)^2},\ b = \frac{\epsilon}{\epsilon-1},\ c = \frac{\epsilon(1-2\epsilon)}{(\epsilon-1)^2}.$$

So,

$$y(t) = ae^{-t/\epsilon} + bte^{-t} + ce^{-t}.$$

After the calculation of $\|y\|_2$ via integration, the parameter ϵ may be calculated through numerical search.

Problem 7.8
From $D^{-1} = (A, B_2, -F, I), \tilde{D}^{-1} = (A, L, -C_2, I)$, we get

$$ND^{-1} = \left[\begin{array}{cc|c} A+B_2F & -B_2F & B_2 \\ 0 & A & B_2 \\ \hline C_2 & 0 & 0 \end{array}\right], \tilde{D}^{-1}\tilde{N} = \left[\begin{array}{cc|c} A & LC_2 & 0 \\ 0 & A+LC_2 & B_2 \\ \hline -C_2 & C_2 & 0 \end{array}\right].$$

Further, by performing similarity transformations w.r.t. $T = \begin{bmatrix} I & I \\ 0 & I \end{bmatrix}$, it is found that $A+B_2F$ is uncontrollable and $A+LC_2$ is unobservable. Then, the result is attained from eliminating them.

It is seen from the A matrcies of D^{-1} and $\tilde{D}^{-1}$ that the zeros of $D(s),\ \tilde{D}(s)$ coincide with the poles of $G_{22}(s)$. Further, the fact that the zeros of $N(s),\ \tilde{N}(s)$ are equal to those of $G_{22}(s)$ follows from

$$\begin{bmatrix} A+B_2F-sI & B_2 \\ C_2 & 0 \end{bmatrix} = \begin{bmatrix} A-sI & B_2 \\ C_2 & 0 \end{bmatrix}\begin{bmatrix} I & F \\ 0 & I \end{bmatrix}$$
$$\begin{bmatrix} A+LC_2-sI & B_2 \\ C_2 & 0 \end{bmatrix} = \begin{bmatrix} I & L \\ 0 & I \end{bmatrix}\begin{bmatrix} A-sI & B_2 \\ C_2 & 0 \end{bmatrix}.$$

Problem 7.9
$Q(s) = (5s+2)/(s+1)$ is derived from the stability of $\hat{e}(s) = S(s)\hat{r}(s) = (1-PQ)/s^2$. So, $K(s) = (s+1)(s+2)(5s+2)/(s^2(s+4))$.

Problem 7.10
(a) $y(\infty) = 0 \Leftrightarrow \hat{y}(s) = P/(1+PK)\hat{d} = P(1-Pq)/s$ must be stable. Then, $1 - P(0)q = 0 \Rightarrow q = 1/P(0)$.
(b) $q = 2 \Rightarrow K(s) = 2(s+2)/s$

Problem 7.11
Omitted.

Problem 7.12
First al all, $\mathcal{F}_\ell(G,\ K) = \mathcal{F}_\ell(\hat{G},\ \hat{K})$ holds and $\hat{D}_{22} = 0$ in $\hat{G}$. Hence, the coefficient matrices of $\mathcal{F}_\ell(\hat{G},\ \hat{K})$ are affine in those of $\hat{K}$.
Once $\hat{K}$ is obtained, K can be computed via $K = (I + \hat{K}D_{22})^{-1}\hat{K}$.

Problem 7.13
By definition, the CLS is internally stable iff

$$\begin{bmatrix} \frac{PK}{1+PK} & \frac{P}{1+PK} \\ \frac{K}{1+PK} & \frac{1}{1+PK} \end{bmatrix} = \begin{bmatrix} PQ & P(1-PQ) \\ Q & 1-PQ \end{bmatrix}$$

is stable. This requires the stability of Q, PQ and $P(1-PQ)$. To enure it, besides the stability of Q, $Q(1) = 0,\ PQ(1) = 1$ are also needed. Such Q is characterized by $Q = \frac{s-1}{s+1}\overline{Q}$ in which $\overline{Q}(s)$ is stable and satisfies $\overline{Q}(1) = 2$.

Problem 7.14
Due to $P(1) = P(\infty) = 0$ and $S = 1-PQ$, $S(1) = S(\infty) = 1$ holds no matter what Q is (these are known as interpolation conditions). According to the maximal modulus theorem in complex analysis, $\|S\|_\infty \geq S(1) = S(\infty) = 1$.

The minimum $\min_K \|S\|_\infty = 1$ is attained at $Q = 0$. The corresponding controller is $K = 0$. This means that doing nothing is the best choice for this performance index. Such a wired solution appears because the performance index $\|S\|_\infty$ is unrealistic, which requires to minimize $|S(j\omega)|$ uniformly over the whole frequency domain and is not possible. More practical goal is to minimize $|S(j\omega)|$ only in the low frequency band because the frequency components of disturbance/reference input concentrate in this band in practice. Refer to Chapter 10 for more detailed discussions.

Problem 7.15
Based on the block diagram, we have

$$\hat{y}_P = P(\hat{u} + \hat{d}), \quad \hat{u} = Q_F\hat{r} + Q_B(P_0\hat{u} - \hat{y}_P).$$

Calculating $\hat{u}$, we get $\hat{u} = Q_F\hat{r} - Q_BP\hat{d}$. So

$$\hat{y}_P = PQ_F\hat{r} + P(1 - PQ_B)\hat{d}.$$

(a) Given the reference model M, it is straightforward to calculate

$$Q_F(s) = \frac{M(s)}{P(s)} = \frac{25(s+1)}{s^2 + 5s + 25}.$$

(b) Substitution of $\hat{d} = 1/s$ and the given Q_B yields (limit ϵ to less than 1)

$$\hat{y}_P = P(1 - PQ_B)\hat{d} = \frac{1}{s+1}\big(1 - \frac{1}{\epsilon s + 1}\big)\frac{1}{s} \Rightarrow y_P(t) = \frac{\epsilon}{1-\epsilon}\Big(e^{-t} - e^{-t/\epsilon}\Big).$$

Then, the solution is the same as Example 7.5: $0 < \epsilon \le 0.151$.

Problem 7.16
Consider the 2nd spec $\|y\|_\infty \le 0.1$. Since the first spec requires $0 < \epsilon \le 0.151$, we need only search in this interval. It is noted that $e^{-t} \ge e^{-t/\epsilon}$ when $\epsilon < 1$. By solving $\dot{y}(t) = 0$ we get a unique solution $t^* = -\ln \epsilon^{\epsilon/(1-\epsilon)}$. So the maximal amplitude is obtained as

$$\|y\|_\infty = \frac{\epsilon}{1-\epsilon}\Big(\epsilon^{\epsilon/(1-\epsilon)} - \epsilon^{1/(1-\epsilon)}\Big).$$

This is an increasing function about ϵ and the solution for $\|y\|_\infty \le 0.1$ can be computed by numerical search.

Chapter 8

Problem 8.1
$\|x\|_2 = 1/2$, $\|y\|_2 = 1/2$, $\langle x, y\rangle_2 = 1/10$. The computation is straightforward.

Problem 8.2
It is noted that $L^{1/2}[D+C(sI-A)^{-1}B]L^{-1/2} = L^{1/2}DL^{-1/2} + L^{1/2}C\cdot(sI-A)^{-1}\cdot BL^{-1/2}$. Invoking the bounded real lemma, statement 1 holds iff there exists $P>0$ satisfying

$$\begin{bmatrix} A^TP+PA & PBL^{-1/2} & (L^{1/2}C)^T \\ (PBL^{-1/2})^T & -\gamma I & (L^{1/2}DL^{-1/2})^T \\ L^{1/2}C & L^{1/2}DL^{-1/2} & -\gamma I \end{bmatrix} < 0.$$

Post-multiplying $L^{1/2}$ to the 2nd column, $L^{-1/2}$ to the 3nd column, then pre-multiplying $L^{1/2}$ to the 2nd row, $L^{-1/2}$ to the 3nd row (this is a congruent transformation) gives the statement 2.
The proof for the converse is done simply by reversing this argument.

Problem 8.3
$G^*(j\omega) + G(j\omega) > 0$ for all frequency ω means that

$$\begin{bmatrix} (j\omega I-A)^{-1}B \\ I \end{bmatrix}^* \begin{bmatrix} 0 & C^T \\ C & D+D^T \end{bmatrix} \begin{bmatrix} (j\omega I-A)^{-1}B \\ I \end{bmatrix} > 0 \ \ \forall\, \omega.$$

Then, application of the strict inequality version of KYP lemma yields the strongly positive real lemma. That, there is a matrix $P>0$ satisfying

$$\begin{bmatrix} A^TP+PA & PB \\ B^TP & 0 \end{bmatrix} - \begin{bmatrix} 0 & C^T \\ C & D+D^T \end{bmatrix} < 0$$

Problem 8.4
Note $G(s-\epsilon) = D + C(sI-(A+\epsilon I))^{-1}B$. For any λ,

$$\begin{aligned} \text{rank} \begin{bmatrix} (A+\epsilon I)-\lambda I & B \end{bmatrix} &= \text{rank} \begin{bmatrix} A-(\lambda-\epsilon)I & B \end{bmatrix} = n \\ \text{rank} \begin{bmatrix} (A+\epsilon I)-\lambda I \\ C \end{bmatrix} &= \text{rank} \begin{bmatrix} A-(\lambda-\epsilon)I \\ C \end{bmatrix} = n \end{aligned}$$

hold owing to the controllability and observability assumption on $(A,\ B,\ C,)$. So, the realization $(A+\epsilon I, B, C, D)$ is also minimal. Further, since A is stable, $A+\epsilon I$ must also be stable for sufficiently small ϵ. Then, the positive real lemma may be applied on $G(s-\epsilon)$ when it is so. And we get

$$\begin{bmatrix} (A+\epsilon I)^T P + P(A+\epsilon I) & PB \\ B^T P & 0 \end{bmatrix} - \begin{bmatrix} 0 & C^T \\ C & D+D^T \end{bmatrix} \leq 0$$
$$\Rightarrow \begin{bmatrix} A^T P + PA & PB \\ B^T P & 0 \end{bmatrix} - \begin{bmatrix} 0 & C^T \\ C & D+D^T \end{bmatrix} \leq -2\epsilon \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix}$$

and $P > 0$. Write $A^T P + PA$ as $-(j\omega - A)^* P - P(j\omega - A)$. Then, multiplying $\begin{bmatrix} (j\omega I - A)^{-1} B \\ I \end{bmatrix}$ and its conjugate transpose to this inequality, we obtain

$$G^*(j\omega) + G(j\omega) \geq 2\epsilon((j\omega T - A)^{-1}B)^* P (j\omega T - A)^{-1} B$$

through some calculation. As $G(s)$ has full normal rank, the right side is positive at finite ω.

Problem 8.5
In the frequency domain,

$$G(j\omega) = \begin{bmatrix} 1 & \frac{1}{\omega^2+1} \\ -\frac{1}{\omega^2+1} & \frac{1}{\omega^2+1} \end{bmatrix} + j \begin{bmatrix} 0 & -\frac{\omega}{\omega^2+1} \\ \frac{\omega}{\omega^2+1} & -\frac{\omega}{\omega^2+1} \end{bmatrix}.$$

So,

$$G^*(j\omega) + G(j\omega) = \begin{bmatrix} 2 & 0 \\ 0 & \frac{2}{\omega^2+1} \end{bmatrix}.$$

It is clear that $G(s)$ is positive real and strictly positive real, but not strongly positive real because the right hand side looses rank at $\omega = \infty$.
Verification in the state space is done by MATLAB numerically.

Problem 8.6
This is simply an analogy of subsection 8.2.2.1. The same storage function $V(x) = x^T P x$ is used. Multiplying to the left and right of the inequality derived in Problem 8.4 with a nonzero vector $\begin{bmatrix} x \\ u \end{bmatrix}$ and its transpose, we have, via the substitution of the dynamics $\dot{x} = Ax + Bu,\ y = Cx + Du$, that

$$\dot{V}(x) \leq 2y^T u - 2\epsilon x^T x < 2y^T u \quad \forall\ x \neq 0. \tag{1.1}$$

Then

$$V(x(t)) < V(x(0)) + 2\int_0^t y^T(\tau) u(\tau) d\tau. \tag{1.2}$$

So, the energy stored in the system is strictly less than the energy supplied by the input.

Chapter 9

Problem 9.1

1. First, there holds

$$AX_1 + RX_2 = X_1 H_- \tag{1.3}$$
$$-QX_1 - A^T X_2 = X_2 H_-. \tag{1.4}$$

(i) Suppose $x \in \text{Ker } (X_1)$. By using $X_2^T X_1 = X_1^T X_2$ and $R \geq 0$, we get $RX_2 x = 0$ from $x^T X_2^T \times(1.3)\times x$. Substituting it into $(1.3)\times x$, we have $X_1 H_- x = 0$. So $\text{Ker}(X_1)$ is H_--invariant.

(ii) When $\text{Ker } (X_1) \neq \{0\}$, due to the invariance $H_-|_{\text{Ker } (X_1)}$ has a stable eigenvalue λ and eigenvector $x \in \text{Ker } (X_1)$ such that $H_- x = \lambda x$. $(1.4)\times x \rightarrow (X_2 x)^*(A + \overline{\lambda} I) = 0$. Since $(A,\ R)$ is stabilizable, this equation and $(X_2 x)^* R = 0$ leads to $X_2 x = 0$, which contradicts the full column rank of $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$.

2. $H \in dom(Ric)$ implies that $A + RX$ is stable. So X is a stabilizing solution. Also, $(A,\ R)$ must be stabilizable.

Problem 9.2

Assume

$$(A - BB^T X)x = \lambda x,\ \Re(\lambda) \geq 0,\ x \neq 0. \tag{1.5}$$

The given Riccati equation can be arranged as

$$(A - BB^T X)^T X + X(A - BB^T X) + XBB^T X + C^T C = 0. \tag{1.6}$$

$x^* \times (1.6) \times x \rightarrow B^T X x = 0,\ Cx = 0$. $Ax = \lambda x$ is obtained from the substitution of $B^T X x = 0$ into (1.5). This contradicts the detectiblility of $(C,\ A)$.

Problem 9.3

Suppose that there are $\Re(\lambda) \geq 0$ and $x \neq 0$ satisfying $x^*[A - BR^{-1}D^T C - \lambda I,\ BR^{-1}B^T] = 0$. Multiplying x to this equation, we have $x^* BR^{-1} B^T x = 0 \rightarrow x^* B = 0$. A further substitution of it into the first equation yields $x^*(A - \lambda I) = 0$, which contradicts the stabilizability of $(A,\ B)$.

Problem 9.4

when D has full column rank, the solvability condition for $Cu + Dv = 0$

w.r.t. v is $Cu \in \text{Ker } D$, i.e., $(I - DR^{-1}D^T)Cu = 0$. In this case, the solution $v = -R^{-1}D^TCu$ is unique. Hence,

$$\begin{bmatrix} A - sI & B \\ C & D \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = 0 \;\Leftrightarrow\; \begin{bmatrix} A - BR^{-1}D^TC - sI \\ (I - DR^{-1}D^T)C \end{bmatrix} u = 0.$$

Chapter 10

Problem 10.1

$$A(s) = \frac{s^2 - 5s + 6}{s^2 + 5s + 6}, \quad G_m(s) = \frac{s(s^2 + 5s + 6)}{(s+5)^2(s^2 + 2s + 5)}.$$

Problem 10.2
The relative degrees are the same because

$$\begin{aligned} C(sI - A - BF)^{-1}B &= C[I - (sI - A)^{-1}BF]^{-1}(sI - A)^{-1}B \\ &= C(sI - A)^{-1}B \cdot [I - F(sI - A)^{-1}B]^{-1} \end{aligned}$$

and $[I - F(sI - A)^{-1}B]^{-1}$ has zero relative degree. Conclusion on the zeros follows from

$$\begin{bmatrix} A + BF - sI & B \\ C + DF & D \end{bmatrix} = \begin{bmatrix} A - sI & B \\ C & D \end{bmatrix} \begin{bmatrix} I & 0 \\ F & I \end{bmatrix}.$$

Problem 10.3
Case 1: The zero z is unstable, but the poles $p_1,\ p_2$ are stable.
In this case, the infimum is

$$\inf \|e\|_2^2 = \frac{2}{|z|}$$

which only depends on the zero and increases as the zero gets close to the origin.
Case 2: $z,\ p_1$ are unstable, p_2 is stable.
In this case, the poles are real and the infimum is

$$\inf \|e\|_2^2 = \frac{2}{|z|} + 8\frac{p_1}{(z - p_1)^2}.$$

It gets larger as $z \to 0$ or $p_1 \to z$.
Case 3: $z,\ p_1 \neq p_2$ are all unstable and real.
The infimum is

$$\begin{aligned} \inf \|e\|_2^2 =& \frac{2}{|z|} + 8\frac{p_1}{(z - p_1)^2}\left|\frac{p_1 + p_2}{p_2 - p_1}\right|^2 + 8\frac{p_2}{(z - p_2)^2}\left|\frac{p_1 + p_2}{p_2 - p_1}\right|^2 \\ &- 32\frac{p_1 p_2}{(p_1 + p_2)(z - p_1)(z - p_2)}\left|\frac{p_1 + p_2}{p_2 - p_1}\right|^2. \end{aligned}$$

It increases when $z \to 0$, or $p_1 \to z$, or $p_2 \to z$, or $p_1 \to p_2$, or $p_1 + p_2 \to 0$.
Case 4: z is unstable and $p_1 = \bar{p}_2$ are unstable.
The infimum is

$$\begin{aligned}\inf \|e\|_2^2 =& \frac{2}{|z|} + 8\frac{\Re(p_1)}{|z-p_1|^2}\left|\frac{\Re(p_2)}{\Im(p_2)}\right|^2 + 8\frac{\Re(p_2)}{(z-p_2)^2}\left|\frac{\Re(p_1)}{\Im(p_1)}\right|^2 \\ &- 16\Re\frac{\Re(p_1)^2\bar{p}_1}{p_1^2\overline{(z-p_1)}(z-p_2)}.\end{aligned}$$

It increases when $z \to 0$, or $p_1 = \bar{p}_2 \to z$, or $p_1 = \bar{p}_2 \to 0$, or $\Im(p_1) = -\Im(p_2) \to 0 \Leftrightarrow p_1 \to p_2$.

Problem 10.4
A realization of the unstable plant is $P(s) = (1, 1, 1, 0)$. So $G_{22} = -P = (1, 1, -1, 0)$. Choosing $f = -2, l = 2$, we have $A_f = A_l = -1$. As a result,

$$\begin{bmatrix} D & -Y \\ N & -X \end{bmatrix} = \begin{bmatrix} \frac{s-1}{s+1} & \frac{4}{s+1} \\ -\frac{1}{s+1} & \frac{s+3}{s+1} \end{bmatrix}.$$

Further, the plant has no unstable zero, so a suboptimal free parameter is $Q(s, \epsilon) = N^{-1}X/(1 + \epsilon s) = (s + 3)/(1 + \epsilon s)$. Then,

$$K(s) = (Y - NQ)(X - DQ)^{-1} = \frac{4(1 + \epsilon s)}{\epsilon s(s + 3)} + \frac{s - 1}{\epsilon s}.$$

Chapter 11

Problem 11.1
The uncertain system is described by

$$P(s) = P_0(s)[1 + W(s)\delta(s)], \quad |\delta(j\omega)| \le 1 \ \forall\ \omega.$$

So $\Delta(s) = P_0(s)W(s)\delta(s)$ and the upper bound of uncertainty gain is characterized by

$$|W(j\omega)| \ge \left|\frac{\Delta(j\omega)}{P_0(j\omega)}\right|, \quad \frac{\Delta(s)}{P_0(s)} = \frac{As^2}{s^2 + s\zeta_1\omega_1 s + \omega^2}.$$

Computing the frequency response of $\Delta(s)/P_0(s)$ w.r.t the vertices of uncertain parameters, then plot them in the same Bode plot, we can find an upper bound by curve fitting with a high-pass transfer function.

Problem 11.2
It is not difficult to get

$$\begin{aligned}\Delta(j\omega) =& P(j\omega) - P_0(j\omega) = \frac{k - k_0 + j\omega(k\tau_0 - \tau k_0)}{(1 + j\tau\omega)(1 + j\tau_0\omega)} \\ \Rightarrow & |\Delta(j\omega)|^2 = \frac{(k - k_0)^2 + \omega^2(k\tau_0 - \tau k_0)^2}{(1 + \tau^2\omega^2)(1 + \tau_0^2\omega^2)}.\end{aligned}$$

At low frequencies, $|\Delta(j\omega)| \approx |k - k_0|$. So $k_0 = (k_{\min} + k_{\max})/2$, the mean value, is a good choice. At high frequencies $|\Delta(j\omega)| \approx |k\tau_0 - \tau k_0|/\tau_0\tau\omega$. Then, the maximum $\tau_{\max}$ is better for τ_0. Further, when $\tau_0 = \tau_{\max}$ the corner frequency of the numerator $k - k_0 + (k\tau_0 - \tau k_0)s$ is bigger that that of $1 + \tau_0 s$ in the denominator so that $|\Delta(j\omega)|$ becomes a decreasing function of ω.
In summary, a good pair of nominal parameters is $k_0 = 1.0,\ \tau_0 = 1.3$, rather than the usually adopted pair of mean values.

Problem 11.3

$$\begin{aligned}&A = \lambda_1 A_1 + \lambda_2 A_2 + \lambda_3 A_3 + \lambda_4 A_4,\ B = \lambda_1 B_1 + \lambda_2 B_2 + \lambda_3 B_3 + \lambda_4 B_4 \\ &A_1 = A_2 = \begin{bmatrix} 0 & 1 \\ 0 & \frac{1}{T_1} \end{bmatrix},\ A_3 = A_4 = \begin{bmatrix} 0 & 1 \\ 0 & \frac{1}{T_2} \end{bmatrix} \\ &B_1 = \begin{bmatrix} 0 \\ \frac{K_1}{T_1} \end{bmatrix},\ B_2 = \begin{bmatrix} 0 \\ \frac{K_2}{T_1} \end{bmatrix},\ B_3 = \begin{bmatrix} 0 \\ \frac{K_1}{T_2} \end{bmatrix},\ B_4 = \begin{bmatrix} 0 \\ \frac{K_2}{T_2} \end{bmatrix}.\end{aligned}$$

Problem 11.4

$$A = \lambda_1 A_1 + \lambda_2 A_2 + \lambda_3 A_3 + \lambda_4 A_4,\ B = \lambda_1 B_1 + \lambda_2 B_2 + \lambda_3 B_3 + \lambda_4 B_4$$

$$A_1 = \begin{bmatrix} 0 & \frac{k_1}{J_1} & 0 \\ -1 & 0 & 1 \\ 0 & -\frac{k_1}{J_M} & 0 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & \frac{k_2}{J_1} & 0 \\ -1 & 0 & 1 \\ 0 & -\frac{k_2}{J_M} & 0 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0 & \frac{k_1}{J_2} & 0 \\ -1 & 0 & 1 \\ 0 & -\frac{k_1}{J_M} & 0 \end{bmatrix},\ A_4 = \begin{bmatrix} 0 & \frac{k_2}{J_2} & 0 \\ -1 & 0 & 1 \\ 0 & -\frac{k_2}{J_M} & 0 \end{bmatrix}$$

$$B_1 = B_2 = \begin{bmatrix} \frac{1}{J_1} \\ 0 \\ 0 \end{bmatrix},\ B_3 = B_4 = \begin{bmatrix} \frac{1}{J_2} \\ 0 \\ 0 \end{bmatrix}.$$

Problem 11.5
(1) Rectangle case: there are 4 vertices

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$

(1) Triangle case: there are 3 vertices. Two are fixed as $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 2, \\ 4 \end{bmatrix}$ and the 3rd one is free. The vertex that minimizes the area of the triangle is given by the intersection point of two tangent lines of function $y = m^2$ passing through, respectively, the points $(m,\ m^2) = (1,\ 1),\ (2,\ 4)$. The solution is $\begin{bmatrix} 3/2 \\ 9/4 \end{bmatrix}$.
The trapezoid with the minimal area is bounded by 4 straight lines: one connecting the two ends of the curve $y = m^2$ (whose gradient is 3), two aforementioned tangent lines, and one with a gradient 3 and intersects the curve $y = m^2$ at only one point. The intersection point is computed by solving $\frac{dm^2}{dm}|_{m_0} = 3 \to m_0 = 3/2$. And the line is given by $y - m_0^2 = 3(m - m_0)$. Solving for the intersection points this line with the other two tangent lines, we finally obtain

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5/4 \\ 25/16 \end{bmatrix}, \begin{bmatrix} 7/4 \\ 49/16 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$

Problem 11.6
The difference from Example 11.6 is: $J_m \neq 1$ and parameter uncertainties are in a multiplicative form. The detail is omitted.

Problem 11.7
Refer to Subsection 20.5.1.

Chapter 12

Problem 12.1
$W(s) = c/(Ms + \mu_0)$

Problem 12.2
Omitted

Problem 12.3
The system cannot be robustly stabilized because $\|WS\|_\infty \geq |WS(j\infty)| = 2 > 1$. This is due to the large uncertainty assumed in the high frequency domain.

Problem 12.4
(a) $a > -4$
(b) $W = a - 10$, $-4 < a < 24$. This range is narrower than the true allowable range and is caused by enlarging the parameter interval into an unrealistic complex disk.

Problem 12.5
(a) $k > 1$
(b) $\hat{e}(s)$ is stable owing to the integrator in the controller. So $e(\infty) = 0$. (c) $k \geq 2$ is obtained based on the stability criterion. Also, the plant can be written as $\tilde{P} = P/(1 + \alpha P)$. If we treat the uncertainty as norm-bounded and apply the robust stability condition $\|PS\|_\infty = 1/(k-1) \leq 1$, we get the same $k \geq 2$.

Problem 12.6
The control objective can be formulated as reducing the norm of the transfer matrix from $(r,\ d)$ to the tracking error e in face of parameter uncertainty. This is equivalent to the robust stabilization of the system in Figure 12.6 where the virtual uncertainties have bounded norms, say, 1. Then, based on the small-gain theorem, a sufficient condition is given by the nominal stability and a norm constraint $\|H_{zw}\|_\infty < 1$ in which $z = (z_1,\ z_2)$, $w = (w_1,\ w_2)$ and the closed loop transfer matrix as well as the corresponding generalized plant G are

$$H_{zw}(s) = \frac{1}{1 + \tilde{P}K} \begin{bmatrix} W_r & -W_r\tilde{P} \\ W_d & -W_d\tilde{P} \end{bmatrix}, \quad G(s) = \begin{bmatrix} W_r & -W_r\tilde{P} & -W_r\tilde{P} \\ W_d & -W_d\tilde{P} & -W_d\tilde{P} \\ 1 & -\tilde{P} & -\tilde{P} \end{bmatrix}.$$

This condition still contains uncertain parameters. However, it can be reduced to conditions at the four vertices of the parameter vector $(M,\ \mu)$. For the detail of the technique, refer to Chapter 18.

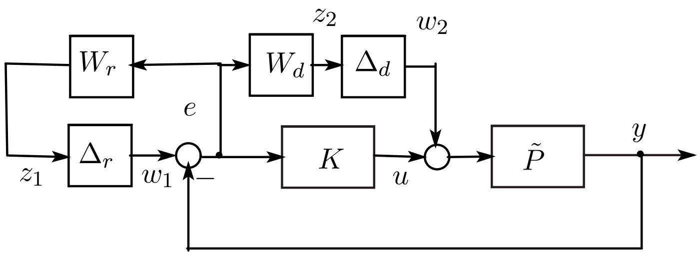


Figure 12.6

Problem 12.7

1. First of all, we note that the nominal stability is ensured by the IMC structure.

P is written as $P_0(1+W\Delta)$ with $\|\Delta\|_\infty \le 1$. Then, the CLS can be transformed equivalently into one with two blocks WP_0Q and Δ. So the robust stability is guaranteed by the norm condition $\|WP_0Q\|_\infty < 1$.

2. The robust performance is equivalent to the robust stability of the CLS in Figure 12.7. From this block diagram, we see that

$$\begin{aligned} z_1 =& W_R(1-P_0Q)(w_1-w_2) \\ z_2 =& WP_0Q(w_1-w_2). \end{aligned}$$

A sufficient condition is obtained as

$$\|H_{zw}\|_\infty < \epsilon, \quad H_{zw} = \begin{bmatrix} W_R(1-P_0Q) \\ WP_0Q \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix}.$$

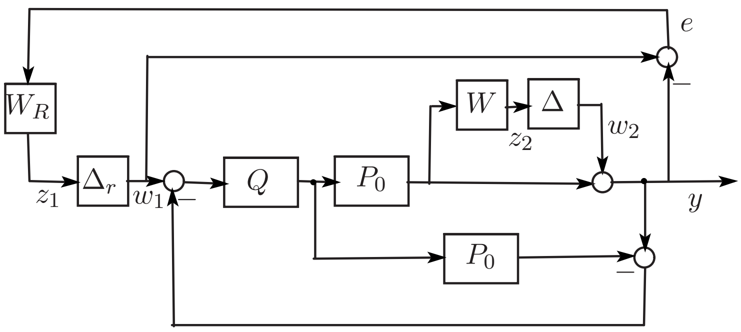


Figure 12.7

Problem 12.8
We try $Q = P^{-1}/(1+\epsilon s) = (s+1)/(1+\epsilon s)$ $(\epsilon > 0)$. Then,

$$\begin{aligned} WP_0Q &= \frac{0.2s}{(0.1s+1)(\epsilon s+1)} \Rightarrow \\ |WP_0Q(j\omega)|^2 &= \frac{(0.2\omega)^2}{(1+(0.1\omega)^2)(1+(\epsilon\omega)^2)} \\ &= \frac{(0.2)^2}{(0.1\epsilon\omega - 1/\omega)^2 + \epsilon^2 + 0.2\epsilon + 0.01}. \end{aligned}$$

So for Spec 1, there must be

$$\begin{aligned} &\|WP_0Q\|_\infty = \frac{0.04}{\epsilon^2 + 0.2\epsilon + 0.01} < 1 \Rightarrow \epsilon^2 + 0.2\epsilon + 0.01 > 0.04 \\ &\Rightarrow \epsilon > 0.1. \end{aligned}$$

The controller has an integrator automatically for the chosen Q. Hence, Spec 2 is also satisfied.
Now, we look at Spec 3.

$$\begin{aligned} \hat{e}(s) = W_R(1 - P_0Q) &= \frac{1}{s}(1 - \frac{1}{1+\epsilon s}) = \frac{1}{s+1/\epsilon} \\ \Rightarrow e(t) = e^{-t/\epsilon} &\Rightarrow \|e\|_2 = \sqrt{\epsilon/2} < 0.5 \Rightarrow \epsilon < 0.5. \end{aligned}$$

Therefore, the range of allowable parameter is $0.1 < \epsilon < 0.5$.

Problem 12.9
1. Name the inputs of $\Delta_n,\ \Delta_m$ as $z_1,\ z_2$ respectively. The CLS may be transformed into one with the dilated uncertainty $[\Delta_n\ \ -\Delta_m]$ and H_{zw}, the transfer matrix from w to $z = (z_1,\ z_2)$ is:

$$H_{zw}(s) = \begin{bmatrix} -\frac{1}{m_0}\frac{K}{1+PK} \\ \frac{1}{m_0}\frac{1}{1+PK} \end{bmatrix} W.$$

So the robust stability condition is $\|H_{zw}\|_\infty < 1$.
2.

$$H_{zw}(s) = \begin{bmatrix} -Q \\ 1 - P_0Q \end{bmatrix} \frac{W}{m_0}.$$

3. The controller $K(s)$ needs an integrator. So, $Q(0) = /P_0(0)$.

4. Set $Q(s) = P^{-1}/(1+\epsilon s)$. Then

$$H_{zw}(s) = \frac{0.2}{1+\epsilon s}\begin{bmatrix} -(s+1) \\ \epsilon s \end{bmatrix} \Rightarrow$$
$$\|H_{zw}(j\omega)\|^2 = 0.4(1+\frac{\omega^2}{1+(\epsilon\omega)^2}) \le 0.4(1+\frac{1}{\epsilon^2}) < 1$$
$$\Rightarrow \epsilon > 1/\sqrt{1.25}.$$

Problem 12.10

For any uncertainty Δ, the sensitivity performance is equivalent to

$$\left| W_1 \frac{1}{1+KP_0(1+W_2\Delta)} \right| < 1 \ \forall\ \omega \Leftrightarrow$$
$$|W_1| < |1+P_0K+W_2P_0K\Delta| \ \forall\ \omega \Leftrightarrow |W_1S_0| < |1+W_2T_0\Delta| \ \forall\ \omega.$$

The right hand side takes its minimum $1-|W_2T_0|$ at the worst-case uncertainty:

$$\Delta : \quad |\Delta(j\omega)| = 1,\ \arg\Delta(j\omega) = -\arg[W_2(j\omega)T_0(j\omega)].$$

Therefore

$$|W_1S_0| < 1-|W_2T_0| \Rightarrow |W_1S_0| + |W_2T_0| < 1 \ \forall\ \omega$$

is necessary and sufficient.

Problem 12.11

1. The CLS is equivalent to one consisting of Δ and WQ_B. So the robust stability condition is $\|WQ_B\| < 1$.
2. The robust disturbance control problem is equivalent to the robust stability problem of the system in Figure 12.11(a) in which $\|\Delta_D\|_\infty \le 1$. Then, from this figure we have

$$z_1 = W_D[w_2 + P_0(w_1+u)],\ z_2 = W(w_1+u),\ u = -Q_B(w_2+P_0w_1)$$
$$\Rightarrow z_1 = W_DP_0(1-P_0Q_B)w_1 + W_D(1-P_0Q_B)w_2,$$
$$z_2 = W(1-P_0Q_B)w_1 - WQ_Bw_2.$$

Hence, the transfer matrix from $(w_1,\ w_2)$ to $(z_1,\ z_2)$ is

$$H_D(s) = \begin{bmatrix} W_DP_0(1-P_0Q_B) & W_D(1-P_0Q_B) \\ W(1-P_0Q_B) & -WQ_B \end{bmatrix}$$

and a robust disturbance control criterion is given by $\|H_D\|_\infty < 1$.

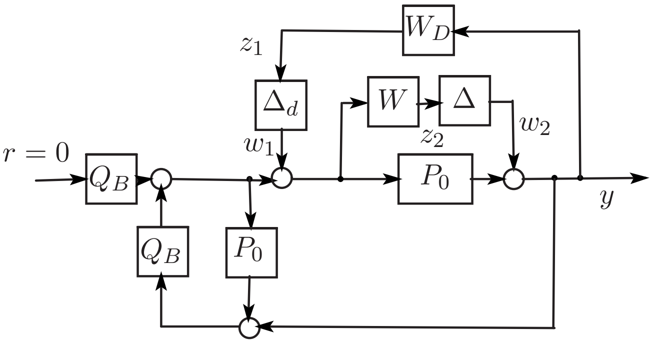


Figure 12.11(a)

3. Similarly, the robust reference tracking problem is equivalent to the robust stability problem of the system in Figure 12.11(b) and $\|\Delta_R\|_\infty \le 1$. Further, the transfer matrix from $(w_1,\ w_2)$ to $(z_1,\ z_2)$ is

$$H_R(s) = \begin{bmatrix} W_R(1 - P_0Q_B) & -W_R(1 - P_0Q_B) \\ WQ_F & -WQ_B \end{bmatrix}$$

and a robust reference tracking criterion is given by $\|H_R\|_\infty < 1$.

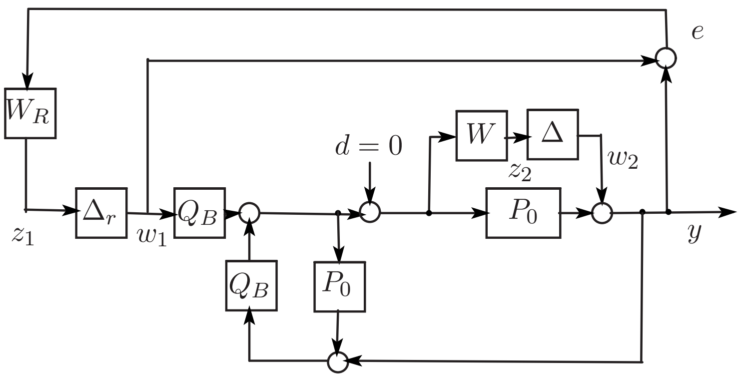


Figure 12.11(b)

Chapter 13

Problem 13.1
The CLS dynamics is $\dot{x} = -\frac{1}{\tau}(1 + kf)x$. By definition, it is quadratically stable if there is a $p > 0$ such that

$$-2\frac{1}{\tau}(1 + kf)p < 0 \Leftrightarrow \frac{1}{\tau}(1 + kf) > 0 \ \forall(\tau,\ k)$$

holds. Since both parameters are positive, this inequality reduces to vertex conditions:

$$1 + k_1 f > 0,\ 1 + k_2 f > 0 \Leftrightarrow f > -\frac{1}{k_2} = -\frac{1}{1.2}.$$

Problem 13.2
Sufficiency: when $\|D\|_2 < 1$, then $\|I - D\Delta\|_2 \geq 1 - \|D\|_2 \|\Delta\|_2 > 1 - 1 \times 1 = 0$. So $I - D\Delta$ is invertible.
Necessity: We make use of the SVD of matrix D:

$$D = U^* \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} V, \quad UU^* = I,\ VV^* = I.$$

$\lambda_1 \geq 1$ if $\|D\|_2 \geq 1$. We prove that the following Δ lowers the rank of $I - D\Delta$. This is clear from

$$\Delta = V^* \begin{bmatrix} \frac{1}{\lambda_1} & & \\ & \ddots & \\ & & 0 \end{bmatrix} U \Rightarrow \|\Delta\|_2 = \frac{1}{\lambda_1} \leq 1$$

$$\Rightarrow I - U^* \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} VV^* \begin{bmatrix} \frac{1}{\lambda_1} & & \\ & \ddots & \\ & & 0 \end{bmatrix} U = U^* \begin{bmatrix} 0 & & \\ & \ddots & \\ & & 1 \end{bmatrix} U.$$

Problem 13.3
The input-output relations are as follows:

$$\begin{aligned} M(s) &: \dot{x} = Ax + Bw,\ z = Cx + Dw \\ \Delta &: w = \Delta z. \end{aligned}$$

According to the bounded real lemma, $\|M\|_\infty <$ implies the existence of a $P > 0$ satisfying

$$\begin{bmatrix} PA + A^T P & PB & C^T \\ B^T P & -I & D^T \\ C & D & -I \end{bmatrix} < 0.$$

Then, multiplication of vector $(x,\ w,\ z)$ and its transpose to this inequality leads to

$$\begin{aligned} 0 >& x^T(PA + A^T P)x + 2x^T PBw + 2x^T C^T z - w^T w + 2w^T D^T z^T - z^T z \\ =& x^T P(Ax + Bw) + (Ax + Bw)^T Px + 2(Cx + Dw)^T z - w^T w - z^T z \\ =& x^T P\dot{x} + \dot{x}^T Px + z^T z - w^T w = \dot{V} + z^T z - w^T w. \end{aligned}$$

Since $w^T w = z^T \Delta^T \Delta z \le z^T z$, we have $\dot{V} < w^T w - z^T z \le 0$ which implies the asymptotic stability of the state $x(t)$.

Problem 13.4
The same problem as Problem 4.12. To be deleted.

Problem 13.5
First of all, the phase angle of $\Delta(j\omega)$ is within $[-\pi/2,\ \pi/2]$ and that of $M(j\omega)$ is in $(-\pi/2,\ \pi/2)$ under the given positive real and strongly positive real properties. So, their sum will never be $\pm 180°$. The CLS is unstable only if the Nyquist plot $M(j\omega)\Delta(j\omega)$ encircles or crosses the critical point $(-1,\ j0)$ for some uncertainty Δ. In the former case, the gain of uncertainty can be reduced so that the Nyqiost plot crosses the critical point. That is, the CLS is not robustly stable iff there is a certain uncertainty and a frequency ω at which

$$1 + M(j\omega)\Delta(j\omega) = 0.$$

This is, however, impossible owing to the phase angle property just stated.

Problem 13.6
Step 1: The transformation is done by setting $z = Cx + Dw$ and $w = \Delta z$.
Step 2: $\|M\|_\infty < 1$ implies that there is a $P > 0$ satisfying

$$\begin{bmatrix} P(A + BF) + (A + BF)^T P & PE & (C + DF)^T \\ E^T P & -I & 0 \\ C + DF & 0 & -I \end{bmatrix} < 0.$$

Step 3: The variable change solution

$$\begin{bmatrix} QA^T + AQ + M^TB^T + BM & E & (CQ+DM)^T \\ E^T & -I & 0 \\ CQ+DM & 0 & -I \end{bmatrix} < 0,\ Q > 0.$$

is derived by setting $Q = P^{-1}$ and $M = FQ$.
Meanwhile, in the variable elimination method we write the condition as

$$\begin{bmatrix} PA + A^TP & PE & C^T \\ E^TP & -I & 0 \\ C & 0 & -I \end{bmatrix} + \begin{bmatrix} PB \\ 0 \\ D \end{bmatrix} F \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix}^T + \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} F^T \begin{bmatrix} PB \\ 0 \\ D \end{bmatrix}^T < 0$$

and compute the following two matrices

$$\begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix}_{\perp}^T = \begin{bmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{bmatrix}, \quad \begin{bmatrix} PB \\ 0 \\ D \end{bmatrix}_{\perp}^T = \begin{bmatrix} P^{-1} & & \\ & 0 & I \\ & I & 0 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} B \\ D \end{bmatrix}_{\perp}^T & 0 \\ 0 & I \end{bmatrix}.$$

Respective multiplication of them onto the preceding inequality provides a solvability condition (another one is included in this condition):

$$\begin{bmatrix} \begin{bmatrix} B \\ D \end{bmatrix}_{\perp}^T & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} AX + XA^T & E & XC^T \\ E^T & -I & 0 \\ CX & 0 & -I \end{bmatrix} \begin{bmatrix} \begin{bmatrix} B \\ D \end{bmatrix}_{\perp}^T & 0 \\ 0 & I \end{bmatrix} < 0.$$

Another condition

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0$$

follows from $P > 0$.

Problem 13.7
Define $z = C_1x + D_{12}u$ and $w = \Delta z$. The parametric system turns into an upper LFT of the uncertainty Δ and the following generalized plant G:

$$\begin{aligned} \dot{x} &= Ax + B_2w + B_1u \\ z &= C_1x + D_{12}u \\ y &= C_2x. \end{aligned}$$

The CLS is quadratically stable if the $\mathcal{H}_\infty$ norm condition

$$\|H_{zw}\|_\infty < 1$$

is satisfied in which H_{zw} is the nominal CLS transfer matrix from w to z. The solution to this $\mathcal{H}_\infty$ problem is provided in Chapter 16.

Problem 13.8
State feedback case:
The CLS is $\dot{x} = (A+BF)x$, which is quadratically stable if there is a $P > 0$ satisfying

$$(A+BF)P + P(A+BF)^T < 0$$

for all uncertain parameters. Defining $M = FP$, this inequality can be reduced to its vertex conditions:

$$A_iP + PA_i + B_iM + M^TB_1^T < 0,\ i = 1,\ 2$$

Output case:
Let the controller be $\dot{x}_K = A_Kx_K + B_Ky,\ u = C_Kx_K + D_Ky$. The CLS is

$$\begin{bmatrix} \dot{x} \\ \dot{x}_K \end{bmatrix} = \begin{bmatrix} A + BD_KC & BC_K \\ B_KC & A_K \end{bmatrix}$$

and is quadratically stable if there is a $P > 0$ such that $A_c^TP + PA_c < 0$ for all λ_1. This can be reduced to the vertex conditions:

$$\begin{bmatrix} A_i + B_iD_KC & B_iC_K \\ B_KC & A_K \end{bmatrix}^T P + P \begin{bmatrix} A_i + B_iD_KC & B_iC_K \\ B_KC & A_K \end{bmatrix},\ i = 1,\ 2.$$

Unfortunately, solution for this problem is still not known.

Chapter 15

Problem 15.1
$\|G\|_2 = \sqrt{7/44}$ (refer to the solution to Problem 2.28 for the detail), $\|y\|_2 = 2\|G\|_2 = \sqrt{7/11}$, $\mathrm{E}[y(\infty)y(\infty)^T] = (3\|G\|_2)^2 = 63/44$.

Problem 15.2
$r(t)$ is the impulse response of integrator $W(s) = 1/s$. So $\|e\|_2 = \|WS\|_2$. Here, the sensitivity is $S = s/(s+k)$. For stability, $k > 0$ is necessary. Then, $k \geq 50$ is computed from $\|WS\|_2 = \sqrt{1/2k} \leq 0.1$.

Problem 15.3
The result is easy to show by starting with $\|G\|_2^2 = \int_0^\infty \mathrm{Tr}(g^T g)dt$. That is, substitution of the impulse response $e(t) = Ce^{At}B$ yields

$$\|G\|_2^2 = \mathrm{Tr}(B^T \int_0^\infty e^{A^T t}C^T Ce^{At}dtB).$$

Then, the conclusion follows from $L_o = \int_0^\infty e^{A^T t}C^T Ce^{At}dt$.

Problem 15.4???
A realization of the unstable plant is $P(s) = (1, 1, 1, 0)$. So $G_{22} = -P = (1, 1, -1, 0)$. Choosing $f = -2, l = 2$, we have $A_f = A_l = -1$. As a result,

$$\begin{bmatrix} D & -Y \\ N & -X \end{bmatrix} = \begin{bmatrix} \frac{s-1}{s+1} & \frac{4}{s+1} \\ -\frac{1}{s+1} & \frac{s+3}{s+1} \end{bmatrix}.$$

Further, the plant has no unstable zero, so a suboptimal free parameter is $Q(s,\epsilon) = N^{-1}X/(1+\epsilon s) = (s+3)/(1+\epsilon s)$. Then,

$$K(s) = (Y - NQ)(X - DQ)^{-1} = \frac{4(1+\epsilon s)}{\epsilon s(s+3)} + \frac{s-1}{\epsilon s}.$$

Problem 15.4
Done by analogy of part 1. Omitted.

Problem 15.5
Done by analogy of part 1. Omitted.

Problem 15.6
This is done by applying a change of input u and output y. It is easy to see

$$\overline{D}_{12} := D_{12}(D_{12}^T D_{12})^{-1/2} \ \Rightarrow\ \overline{D}_{12}^T \overline{D}_{12} = (D_{12}^T D_{12})^{-1/2} \cdot D_{12}^T D_{12} \cdot (D_{12}^T D_{12})^{-1/2} = I$$
$$\overline{D}_{21} := (D_{21} D_{21}^T)^{-1/2} D_{21} \ \Rightarrow\ \overline{D}_{21} \overline{D}_{21}^T = I.$$

Then, we define new input and output as $\bar{u}$, $\bar{y}$ below:

$$\bar{u} := (D_{12}^T D_{12})^{1/2} u, \quad \bar{y} := (D_{21} D_{21}^T)^{-1/2} y$$

and absorb the scaling matrices into the generalized plant and the controller. The transformed system becomes

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 (D_{12}^T D_{12})^{-1/2} \bar{u} \\ z &= C_1 x + D_{11} w + \overline{D}_{12} \bar{u} \\ \bar{y} &= (D_{21} D_{21}^T)^{-1/2} C_2 x + \overline{D}_{21} w \\ \bar{u} &= \overline{K} \bar{u} \\ \overline{K} &= (D_{12}^T D_{12})^{1/2} K (D_{21} D_{21}^T)^{1/2}. \end{aligned}$$

After the design of $\overline{K}(s)$, the real controller $K(s)$ is computed by

$$K(s) = (D_{12}^T D_{12})^{-1/2} \overline{K} (D_{21} D_{21}^T)^{-1/2}.$$

Problem 15.7
Under the given condition, $D_{12}^\dagger = D_{12}^T$ and $D_{21}^\dagger = D_{21}^T$ hold. According to Lemma 2.3, the solvability condition is $D_{12} D_{12}^T D_{11} D_{21}^T D_{21} = D_{11}$ and a solution is $D_K = -D_{12}^T D_{11} D_{21}^T$. Further, $\hat{D}_{11} = D_c = 0$ and $K(s) = \hat{K}(s) + D_K$.

Problem 15.8
Since $C_2 = I$ and $D_{21} = 0$, there hold $G_f = V = (A + L,\ B_1,\ I,\ 0)$. So, $H_{zw} = G_c B_1 + U(Q - F_2)V$. The rest is similar to the output case and omitted.

Problem 15.9
The CLS is

$$\begin{bmatrix} \dot{x} \\ z \end{bmatrix} = \begin{bmatrix} A + B_2 F & B_1 \\ C_1 + D_{12} F & D_{11} \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix}.$$

Applying the state space conditions of $\mathcal{H}_2$ norm given in Section 15.5, we obtain

$$\begin{bmatrix} P(A+B_2F)+(A+B_2F)^TP & (C_1+D_{12}F)^T \\ C_1+D_{12}F & -I \end{bmatrix} < 0$$

$$\begin{bmatrix} W & B_1^TP \\ PB_1 & P \end{bmatrix} > 0, \quad \mathrm{Tr}(W) < \gamma^2.$$

After transforming them into inequalities about $Q = P^{-1}$, we obtain

$$\begin{bmatrix} AQ+QA^T+B_2M+M^TB_2^T & QC_1^T+M^TD_{12}^T \\ C_1Q+D_{12}M & -I \end{bmatrix} < 0$$

$$\begin{bmatrix} W & B_1^T \\ B_1 & Q \end{bmatrix} > 0, \quad \mathrm{Tr}(W) < \gamma^2.$$

by changing the unknown F to a new variable $M = FQ$.

Chapter 16

Problem 16.1
(a) Omitted,

(b) $\|P\|_\infty = |P(j\infty)| = 1$

(c) 1/2

Problem 16.2

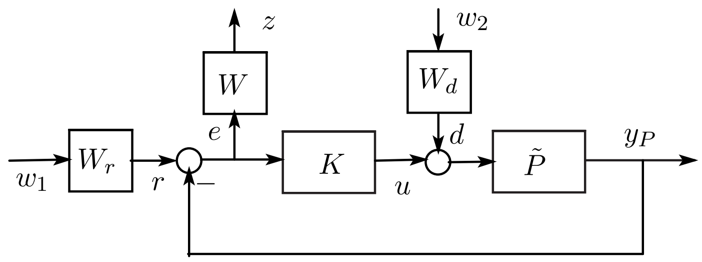


Figure 16.2

First, the robust reference tracking and disturbance suppression performance can be formulated as a minimization problem of the transfer matrix from $(r,\ d)$ to the tracking error e. After inserting the models of the speed reference and wind disturbance, we shift their common part (unstable in particular) such as integrator to the error port (Figure 16.2). Then, the problem boils down to a norm constraint $\|H_{zw}\|_\infty < 1$ in which $w = (w_1,\ w_2)$. Let the state equation of each block be described as

$$\begin{aligned}
\dot{x} &= -\frac{\mu}{M}x + \frac{1}{M}(u+d), \quad y_P = x \\
\dot{x}_W &= A_W x_W + B_W e, \quad z = C_W x_W + D_W e \\
\dot{x}_r &= A_r x_r + B_r w_1, \quad r = C_r x_r + D_r w_1 \\
\dot{x}_d &= A_d x_d + B_d w_2, \quad d = C_d x_d + D_d w_2.
\end{aligned}$$

The measured output y is the tracking error $e = r - y_P = -x + C_r x_r + D_r w_1$.

Then, the state equation of the generalized plant becomes

$$
\begin{bmatrix} \dot{x} \\ \dot{x}_W \\ \dot{x}_r \\ \dot{x}_d \end{bmatrix} = \begin{bmatrix} -\frac{\mu}{M} & 0 & 0 & \frac{1}{M}C_d \\ -B_W & A_W & B_W C_r & 0 \\ 0 & 0 & A_r & 0 \\ 0 & 0 & 0 & A_d \end{bmatrix} \begin{bmatrix} x \\ x_W \\ x_r \\ x_d \end{bmatrix}
+ \begin{bmatrix} 0 & \frac{1}{M}D_d \\ B_W D_r & 0 \\ B_r & 0 \\ 0 & B_d \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{M} \\ 0 \\ 0 \\ 0 \end{bmatrix} u
$$

$$
z = [-D_W \ \ C_W \ \ D_W C_r \ \ 0] \begin{bmatrix} x \\ x_W \\ x_r \\ x_d \end{bmatrix} + [D_W D_r \ \ 0] \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}
$$

$$
y = [-1 \ \ 0 \ \ C_r \ \ 0] \begin{bmatrix} x \\ x_W \\ x_r \\ x_d \end{bmatrix} + [D_r \ \ 0] \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}.
$$

The following three matrices depend on the uncertain parameter vector $\theta = (M,\ \mu)$:

$$
A(\theta) = \begin{bmatrix} -\frac{\mu}{M} & 0 & 0 & \frac{1}{M}C_d \\ -B_W & A_W & B_W C_r & 0 \\ 0 & 0 & A_r & 0 \\ 0 & 0 & 0 & A_d \end{bmatrix}
$$

$$
B_1(\theta) = \begin{bmatrix} 0 & \frac{1}{M}D_d \\ B_W D_r & 0 \\ B_r & 0 \\ 0 & B_d \end{bmatrix}, \quad B_2(\theta) = \begin{bmatrix} \frac{1}{M} \\ 0 \\ 0 \\ 0 \end{bmatrix}.
$$

As a result, in the coefficient matrices of the closed loop system, $A_c(\theta)$ and $B_c(\theta)$ become affine functions of the parameter vector θ (refer to Eqs. (7.43), (7.44)). Further, by the bounded-real lemma the norm condition is satisfied iff there is a matrix $P >$ such that

$$
\begin{bmatrix} A_c(\theta)^T P + P A_c(\theta) & P B_c(\theta) & C_c^T \\ B_c(\theta)^T P & -I & D_c^T \\ C_c & D_c & -I \end{bmatrix} < 0.
$$

This condition can be reduced equivalently to the vertix conditions:

$$\begin{bmatrix} A_c(\theta_i)^T P + PA_c(\theta_i) & PB_c(\theta_i) & C_c^T \\ B_c(\theta_i)^T P & -I & D_c^T \\ C_c & D_c & -I \end{bmatrix} < 0,\ i = 1, \dots, 4$$

in which

$$\theta_1 = \begin{bmatrix} M_1 \\ \mu_1 \end{bmatrix},\ \theta_2 = \begin{bmatrix} M_2 \\ \mu_1 \end{bmatrix},\ \theta_1 = \begin{bmatrix} M_1 \\ \mu_2 \end{bmatrix},\ \theta_1 = \begin{bmatrix} M_2 \\ \mu_2 \end{bmatrix}.$$

Additional problem:
Consider the quadratic stabilization. The input may be set as a state feedback $u = fx$. Then, for all uncertain parameters under consideration, the quadratic stability condition

$$(-\frac{\mu}{M} + f\frac{1}{M})P + P(-\frac{\mu}{M} + f\frac{1}{M}) < 0, \quad P > 0$$

falls down to the following vertex conditions:

$$(-\frac{\mu_i}{M_j} + f\frac{1}{M_j})P < 0,\ P > 0 \quad i,\ j = 1,\ 2.$$

This further reduces to

$$f < \mu_1.$$

Problem 16.3
The stability of H_{yw} requires $\to PQ(0) = 1 \to b = 1$. Then, $\|H_{yw}\|_\infty = |H_{yw}(0)| = a \to 0 < a < 1$. $K(s) = (s+1)/(as)$. Smaller a contributes more to the disturbance suppression, but results in bigger control input.

Problem 16.4
The CLS is

$$\begin{bmatrix} \dot{x} \\ z \end{bmatrix} = \begin{bmatrix} A + B_2 F & B_1 \\ C_1 + D_{12}F & D_{11} \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix}.$$

Applying the bounded real lemma to this CLS, the problem has a solution iff there is a $P > 0$ satisfying

$$\begin{bmatrix} A^T P + PA & PB_1 & C_1^T \\ B_1^T P & -\gamma I & D_{11}^T \\ C_1 & D_{11} & -\gamma I \end{bmatrix} + \begin{bmatrix} PB_2 \\ 0 \\ D_{12} \end{bmatrix} F \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix}^T + \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} F^T \begin{bmatrix} PB_2 \\ 0 \\ D_{12} \end{bmatrix}^T$$

$< 0.$

Setting $X = P^{-1}$ and noting

$$[B_2^T P \;\; 0 \;\; D_{12}^T]_\perp = \begin{bmatrix} X & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{bmatrix} \begin{bmatrix} N_X & 0 \\ 0 & I \end{bmatrix}, \; [I \;\; 0 \;\; 0]_\perp = \begin{bmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{bmatrix},$$

we arrive at the solvability condition

$$\begin{bmatrix} N_X^T & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} AX + XA^T & XC_1^T & B_1 \\ C_1 X & -\gamma I & D_{11} \\ B_1^T & D_{11}^T & -\gamma I \end{bmatrix} \begin{bmatrix} N_X & 0 \\ 0 & I \end{bmatrix} < 0.$$

Another condition $\begin{bmatrix} -\gamma I & D_{11}^T \\ D_{11} & -\gamma I \end{bmatrix} < 0$ in bounded real lemma is contained in this LMI.

Problem 16.5
The multiplication yields

$$\begin{bmatrix} \Pi_1^T A_c^T P \Pi_1 + \Pi_1^T P A_c \Pi_1 & \Pi_1^T P B_c & \Pi_1^T C_c^T \\ B_c^T P \Pi_1 & -\gamma I & D_c^T \\ C_c \Pi_1 & D_c & -\gamma I \end{bmatrix} < 0.$$

The LMI solution follows from a substitution of

$$\Pi_1^T P A_c \Pi_1 = \begin{bmatrix} AX + B_2 \mathbb{C} & A + B_2 \mathbb{D} C_2 \\ \mathbb{A} & YA + \mathbb{B} C_2 \end{bmatrix}, \quad \Pi_1^T P B_c = \begin{bmatrix} B_1 + B_2 \mathbb{D} D_{21} \\ Y B_1 + \mathbb{B} D_{21} \end{bmatrix}$$
$$C_c \Pi_1 = \begin{bmatrix} C_1 X + D_{12} \mathbb{C} & C_1 + D_{12} \mathbb{D} C_2 \end{bmatrix}.$$

Problem 16.6
The only difference lies in the changes of $-\gamma I \to -\gamma L$, $-\gamma I \to -\gamma L^{-1}$ in the lower right corner. So the derivation is the same as Subsection 16.3.1, and the matrix inverse L^{-1} is set as J.

Problem 16.7
After the treatment as in Problem 16.5, the solvability condition for the scaled $\mathcal{H}_\infty$ control problem becomes $P > 0$ and

$$\begin{bmatrix} \Pi_1^T A_c^T P \Pi_1 + \Pi_1^T P A_c \Pi_1 & \Pi_1^T P B_c & \Pi_1^T C_c^T \\ B_c^T P \Pi_1 & -\gamma L & D_c^T \\ C_c \Pi_1 & D_c & -\gamma J \end{bmatrix} < 0.$$

Then, substitution of the matrix blocks calculated there leads to

$$\mathrm{He}\begin{bmatrix} AX + B_2\mathbb{C} & A + B_2\mathbb{D}C_2 & B_1 + B_2\mathbb{D}D_{21} & 0 \\ \mathbb{A} & YA + \mathbb{B}C_2 & YB_1 + \mathbb{B}D_{21} & 0 \\ 0 & 0 & -\frac{\gamma}{2}L & 0 \\ C_1X + D_{12}\mathbb{C} & C_1 + D_{12}\mathbb{D}C_2 & D_{11} + D_{12}\mathbb{D}D_{21} & -\frac{\gamma}{2}J \end{bmatrix} < 0 \quad (1.7)$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0 \quad (1.8)$$

$$LJ = I. \quad (1.9)$$

Here, the same variable changes are used.
The same K-L iteration procedure in Section 16.7 may be applied.

Chapter 17

Problem 17.1
Necessity: $\det[I - M(j\omega)\Delta(j\omega)] = 0$ at some ω implies that the CLS is unstable w.r.t. this $\Delta(s)$.
Sufficiency: First, we fix the dynamics of uncertainty $\Delta(s)$, only let its gain vary from zero to the allowable bound. When the gain is zero, we have $\det(I - M\Delta) = 1$. When the gain is raised the trajectory of $\det[I - M(j\omega)\Delta(j\omega)]$ expands continuously outward. So, it must intersect the origin before encircling the origin. Therefore, the CLS is stable whenever $\det[I - M(j\omega)\Delta(j\omega)] \neq 0$ for all frequencies. As the dynamics of $\Delta(s)$ in the proof is arbitrary, the conclusion is true for any $\Delta(s)$ in the given class.

Problem 17.2
(a) $\mu_\Delta(M(j\omega_0)) \geq 1/\gamma$ implies that there is some $\gamma_0 < \gamma$ such that $\mu_\Delta(M(j\omega_0)) = 1/\gamma_0$ because $\mu_\Delta(M(j\omega_0))$ is bounded. That is, there is a complex matrix $\Delta_0(j\omega_0)$ satisfying

$$\sigma_{\max}[\Delta_0(j\omega_0)] = \frac{1}{\gamma_0}, \quad \det[I - M(j\omega_0)\Delta_0(j\omega_0)] = 0$$

by the definition of μ. Then, there is a vector u such that $[I - M(j\omega_0)\Delta_0(j\omega_0)]$ $u = 0$.
(b) is trivial.
(c) and (d) can be done by using the algorithm given in the proof of small-gain theorem.
(e) The $\Delta(s)$ constructed in (c) and (d) belongs to the given class and satisfies $\det[I - M(j\omega_0)\Delta(j\omega_0)] = 0$ so that $j\omega_0$ is a pole of the CLS. Therefore, for the robust stability $\mu_\Delta(M(j\omega)) \leq 1/\gamma$ must hold in the whole frequency domain for all uncertainty in the given class.

Chapter 18

Problem 18.1
To ensure that the CLS coefficient matrices are affine in the uncertain parameter vector, we may add two low-pass filters to the input and output ports of the controller:

$$K(s) = F_1(s)\overline{K}(s)F_2(s).$$

Let the input and output of the new controller $\overline{K}(s)$ be denoted by $(\overline{y},\ \overline{u})$, then

$$\overline{y} = F_2(s)y, \quad u = F_1(s)\overline{u}.$$

Next, we absorb these filters into the generalized plant $G(s)$ in the controller design:

$$\overline{G}(s) = \begin{bmatrix} I & 0 \\ 0 & F_2 \end{bmatrix} G \begin{bmatrix} I & 0 \\ 0 & F_1 \end{bmatrix}.$$

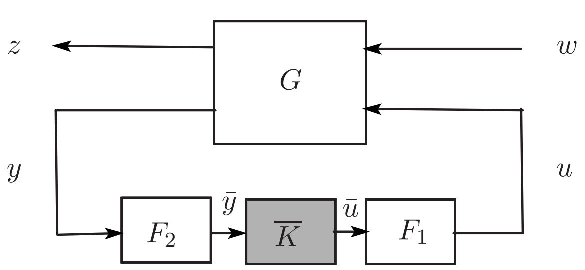
Specifically, when the filters are selected as 1st order with break frequencies $(\omega_1,\ \omega_2)$

$$F_i(s) = \frac{\omega_i}{s+\omega_i} I,$$

we have

$$\overline{G}(s) = \left[\begin{array}{ccc|c:c} -\omega_2 I & C_2 & 0 & D_{21} & 0 \\ 0 & A & B_2 & B_1 & 0 \\ 0 & 0 & -\omega_1 I & 0 & \omega_1 I \\ \hline 0 & C_1 & D_{12} & D_{11} & 0 \\ \hdashline \omega_2 I & 0 & 0 & 0 & 0 \end{array}\right]$$

The computation is based on the cascade connection formula of Subsection 4.1.9. Then, the approach of Section 18.3 can be applied.



Problem 18.2
(a) Refer to the solution to Problem 13.3.

(b) We note that, by Schur's lemma, (18.35) implies

$$\begin{aligned} 0 >& A^T P + PA - \begin{bmatrix} PB & C^T \end{bmatrix} (-I)^{-1} \begin{bmatrix} B^T P \\ C \end{bmatrix} \\ =& A^T P + PA + PBB^T P + C^T C. \end{aligned}$$

Next, via a completion of square, we get ($\because\ \Delta^T \Delta \leq I$)

$$\begin{aligned} & A^T P + PA + C^T \Delta^T B^T P + PB\Delta C \\ &= A^T P + PA + PBB^T P + C^T \Delta^T \Delta C - (PB - C^T \Delta^T)(B^T P - \Delta C) \\ &\leq A^T P + PA + PBB^T P + C^T \Delta^T \Delta C \\ &\leq A^T P + PA + PBB^T P + C^T C. \end{aligned}$$

So, the quadratic stability is guaranteed by (18.35).

Chapter 19

Problem 19.1
The pole location $-h_2 < \Re(z) = (z+\bar{z})/2 < -h_1$ requires a matrix $X > 0$ satisfying

$$XA + T^T X + h_1 X < 0, \quad XA + T^T X + h_2 X > 0$$

simultaneously by following the variable replacement rule of (19.17).

Problem 19.2
Shifting the imaginary axis of the s-plane to $s = a$ and let the resultant coordinate be p-plane. Then, any point z is the s-plane becomes $p = z - a$ in the p-plane. After this transformation, the given sector turns into a sector whose vertex is the origin $p = 0$ in the new plane and the known result may be applied. So,

$$\begin{aligned} f_D(z) &= p \begin{bmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{bmatrix} + \overline{p} \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix} < 0 \\ &= z \begin{bmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{bmatrix} + \overline{z} \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix} - 2a \begin{bmatrix} \sin\theta & 0 \\ 0 & \sin\theta \end{bmatrix} < 0. \end{aligned}$$

Problem 19.3

Noting that $I - \Delta^T\Delta \geq 0$, the inequality follows from

$$\begin{aligned} &\mathcal{Y}^T[(Q\otimes I) - (Q\otimes \Delta^T\Delta)]\mathcal{Y} = \mathcal{Y}^T[Q\otimes(I-\Delta^T\Delta)]\mathcal{Y} \\ &= \sum_{i=1}^{k}\sum_{j=1}^{k} q_{ij}\mathcal{Y}_i^T(I-\Delta^T\Delta)\mathcal{Y}_j \\ &= \sum_{i=1}^{k}\sum_{j=1}^{k} q_{ij}[(I-\Delta^T\Delta)^{1/2}\mathcal{Y}_i]^T(I-\Delta^T\Delta)^{1/2}\mathcal{Y}_j \\ &= \{[I\otimes(I-\Delta^T\Delta)^{1/2}]\mathcal{Y}\}^T(Q\otimes I)\{[I\otimes(I-\Delta^T\Delta)^{1/2}]\mathcal{Y}\} \\ &\geq 0. \end{aligned}$$

Problem 19.4
We note that $\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x} = x^T(A^T(t)P + PA(t))x$. Multiplying

x and x^T to the (i,j) block of $N_D(A(t),P)$, we see that

$$\begin{aligned}
&x^T N_D(A(t),P)_{ij}x = x^T[l_{ij}P + m_{ij}PA(t) + m_{ji}A^T(t)P]x \\
&= l_{ij}x^T Px + m_{ij}x^T PA(t)x + m_{ji}(A(t)x)^T Px \\
&= l_{ij}V(x) + \frac{1}{2}m_{ij}x^T[PA(t) + A^T(t)P]x + \frac{1}{2}m_{ji}x^T[PA(t) + A^T(t)P]x \\
&= l_{ij}V(x) + \frac{1}{2}m_{ij}\dot{V}(x) + \frac{1}{2}m_{ji}\dot{V}(x) \\
&= V(x)[l_{ij} + \frac{1}{2}m_{ij}\frac{\dot{V}(x)}{V(x)} + \frac{1}{2}m_{ji}\frac{\dot{V}(x)}{V(x)}].
\end{aligned}$$

(The 3rd equation is due to $x^T PA(t)x = x^T PA(t)^T x$ as it is a scalar.) So, by multiplying all blocks of the given matrix inequality with x and x^T, we get

$$\begin{aligned}
0 &> (I \otimes x^T)N_D(A(t),P)(I \otimes x) \\
&= V(x)(L + \frac{1}{2}\frac{\dot{V}(x)}{V(x)}M + \frac{1}{2}\frac{\dot{V}(x)}{V(x)}M^T) \\
&= V(x) \cdot f_D(\frac{1}{2}\frac{\dot{V}(x)}{V(x)}).
\end{aligned}$$

Dividing this inequality with $V(x) > 0$, we see that the characteristic function satisfies $f_D(\frac{1}{2}\frac{\dot{V}(x)}{V(x)}) < 0$, i.e., $\frac{1}{2}\frac{\dot{V}(x)}{V(x)} \in D$. Since $\frac{1}{2}\frac{\dot{V}(x)}{V(x)}$ takes real values, it must be bounded by the two intersection points of the region D and the real axis. Denote these two points by $(-a,\ j0)$ and $(-b,\ j0)$. Then

$$-b \leq \frac{1}{2}\frac{\dot{V}(x)}{V(x)} \leq -a \Leftrightarrow -2bV(x) \leq \dot{V}(x) \leq -2aV(x).$$

As the solution of $\dot{y} = -2py$ is $y(t) = e^{-2pt}y(0)$, according to the well-known comparison principle[**?**]

$$e^{-2bt}V(0) \leq V(x) \leq e^{-2at}V(0)$$

holds. Further, noting that $\lambda_{\min}(P)\|x\|_2^2 \leq x^T Px \leq \lambda_{\max}(P)\|x\|_2^2$, we arrive at

$$\begin{aligned}
&\lambda_{\min}(P)\|x(t)\|_2^2 \leq e^{-2at}V(0), \quad \lambda_{\max}(P)\|x(t)\|_2^2 \geq e^{-2bt}V(0) \\
&\Rightarrow c_1 e^{-bt} \leq \|x(t)\| \leq c_2 e^{-at}.
\end{aligned}$$

This means that the state response is bounded by two exponentially convergent functions with convergence rates a and b.

Problem 19.5
1. The matrices in the characteristic function are

$$L = \begin{bmatrix} -r & c \\ c & -r \end{bmatrix}, \ M = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = M_1 M_2^T \Rightarrow M_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ M_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

2.
$$L = 2\sigma, \quad M = 1 \Rightarrow M_1 = M_2 = 1.$$

3.
$$L = 0, \ M = \begin{bmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{bmatrix} \Rightarrow M_1 = M, \ M_2 = I_2.$$

Expansion of (19.59) is done by the substitution of these matrices and the matrices given after (19.60). The computation is straightforward and omitted here.

Chapter 20

Problem 20.1
The following factorization as well as the solution for $P > 0$ is due to Lemma 3.1

$$MM^T = Y - X^{-1}, \quad P = \begin{bmatrix} Y & M \\ M^T & I \end{bmatrix}.$$

Meanwhile, the result about the scaling matrix L is from a variation of Lemma 3.1, derived from the $(2,\ 2)$ block of $L > 0$:

$$N^T N = L_3 - J_3^{-1}, \quad L = \begin{bmatrix} I & N \\ N^T & L_3 \end{bmatrix}.$$

Step 3 is almost the same as Subsection 16.3.1, the difference lies in the existence of L_a and its inverse J_a in (20.38). The computation is based on the solutions to Problems 16.6 and 8.2.

Problem 20.2
The CLS has the following coefficient matrices:

$$\begin{aligned}
A_c(p) &= \begin{bmatrix} A(p) + B_2(p)D_K(p)C_2(p) & B_2(p)C_K(p) \\ B_K(p)C_2(p) & A_K(p) \end{bmatrix} \\
B_c(p) &= \begin{bmatrix} B_1(p) + B_2(p)D_K(p)D_{21} \\ B_K(p)D_{21} \end{bmatrix} \\
C_c(p) &= \begin{bmatrix} C_1(p) + D_{12}D_K(p)C_2(p) & D_{12}C_K(p) \end{bmatrix} \\
D_c(p) &= D_{11} + D_{12}D_K(p)D_{21}.
\end{aligned}$$

The key point is how to avoid nonlinear terms of uncertain parameters appearing in these matrices. So, from the structure of these matrices we have the following:

1. When $B_2(p),\ C_2(p)$ both depend on p, to ensure linearity about p, D_K must be zero and $B_K,\ C_K$ must be constant matrices.

In this case

$$
\begin{aligned}
\mathbb{A}(p) =& N(A_{K0} + \sum p_i A_{Ki})M^T + NB_K(C_{20} + \sum p_i C_{2i})X \\
& + Y(B_{20} + \sum p_i B_{2i})C_K M^T + Y(A_0 + \sum p_i A_i)X \\
\Rightarrow & \mathbb{A}_0 = NA_{K0}M^T + NB_K C_{20}X + YB_{20}C_K M^T + YA_0 X \\
& \mathbb{A}_i = NA_{Ki}M^T + NB_K C_{2i}X + YB_{2i}C_K M^T + YA_i X \\
\Rightarrow & A_{K0} = N^\dagger\{\mathbb{A}_0 - (NB_K C_{20}X + YB_{20}C_K M^T + YA_0 X)\}(M^\dagger)^T \\
& A_{Ki} = N^\dagger\{\mathbb{A}_i - (NB_K C_{2i}X + YB_{2i}C_K M^T + YA_i X)\}(M^\dagger)^T \\
\mathbb{B}(p) =& NB_K = \mathbb{B} \Rightarrow B_K = N^\dagger \mathbb{B} \\
\mathbb{C}(p) =& C_K M^T = \mathbb{C} \Rightarrow C_K = \mathbb{C}(M^\dagger)^T
\end{aligned}
$$

2. When both B_2 and C_2 are constant matrices, all matrices $A_K(p),\ B_K(p),\ C_K(p),\ D_K(p)$ may be affine in p.

3. When only B_2 is constant, then B_K and D_K must be constant matrices. In this case

$$
\begin{aligned}
& \mathbb{A}(p) = N(A_{K0} + \sum p_i A_{Ki})M^T + NB_K(C_{20} + \sum p_i C_{2i})X \\
& + YB_2(C_{K0} + \sum p_i C_i)M^T + Y[A_0 + \sum p_i A_i + B_2 D_K(C_{20} + \sum p_i C_{2i})]X \\
& \Rightarrow \mathbb{A}_0 = NA_{K0}M^T + NB_K C_{20}X + YB_2 C_{K0}M^T + Y(A_0 + B_2 D_K C_{20})X \\
& \mathbb{A}_i = NA_{Ki}M^T + NB_K C_{2i}X + YB_2 C_{Ki}M^T + Y(A_i + B_2 D_K C_{2i})X \\
& \Rightarrow A_{K0} = N^\dagger\{\mathbb{A}_0 - [NB_K C_{20}X + YB_2 C_{K0}M^T + Y(A_0 + B_2 D_K C_{20})X]\}(M^\dagger)^T \\
& A_{Ki} = N^\dagger\{\mathbb{A}_i - [NB_K C_{2i}X + YB_2 C_{Ki}M^T + Y(A_i + B_2 D_K C_{2i})X]\}(M^\dagger)^T \\
& \mathbb{B}(p) = NB_K + YB_2 D_K = \mathbb{B} \Rightarrow B_K = N^\dagger(\mathbb{B} - YB_2 D_K) \\
& \mathbb{C}(p) = (C_{K0} + \sum p_i C_{Ki})M^T + D_K(C_{20} + \sum p_i C_{2i})X \\
& \Rightarrow \mathbb{C}_0 = C_{K0}M^T + D_K C_{20}X,\ \mathbb{C}_i = C_{Ki}M^T + D_K C_{2i}X \\
& \Rightarrow C_{K0} = (\mathbb{C}_0 - D_K C_{20}X)(M^\dagger)^T,\ C_{Ki} = (\mathbb{C}_i - D_K C_{2i}X)(M^\dagger)^T \\
& \mathbb{D}(p) = D_K = \mathbb{D}_0
\end{aligned}
$$

4. When only C_2 is constant, then C_K and D_K must be constant matrices.

In this case

$$\mathbb{A}(p) = N(A_{K0} + \sum p_i A_{Ki})M^T + N(B_{K0} + \sum p_i B_{Ki})C_2 X$$
$$+ Y(B_{20} + \sum p_i B_{2i})C_2 M^T + Y[A_0 + \sum p_i A_i + (B_{20} + \sum p_i B_{2i})D_K C_2]X$$
$$\Rightarrow \mathbb{A}_0 = NA_{K0}M^T + NB_{K0}C_2X + YB_{20}C_KM^T + Y(A_0 + B_{20}D_KC_2)X$$
$$\mathbb{A}_i = NA_{Ki}M^T + NB_{Ki}C_2X + YB_{2i}C_KM^T + Y(A_0 + B_{2i}D_KC_2)X$$
$$\Rightarrow A_{K0} = N^\dagger\{\mathbb{A}_0 - (NB_{K0}C_2X + YB_{20}C_KM^T + Y(A_0 + B_{20}D_KC_2)X)\}(M^\dagger)^T$$
$$A_{Ki} = N^\dagger\{\mathbb{A}_i - (NB_{Ki}C_2X + YB_{2i}C_KM^T + Y(A_0 + B_{2i}D_KC_2)X)\}(M^\dagger)^T$$
$$\mathbb{B}(p) = N(B_{K0} + \sum p_i B_{Ki}) + Y(B_{20} + \sum p_i B_{2i})D_K$$
$$\Rightarrow \mathbb{B}_0 = NB_{K0} + YB_{20}D_K,\ \mathbb{B}_i = NB_{Ki} + YB_{2i}D_K$$
$$\Rightarrow B_{K0} = N^\dagger(\mathbb{B}_0 - YB_{20}D_K),\ B_{Ki} = N^\dagger(\mathbb{B}_i - YB_{2i}D_K)$$
$$\mathbb{C}(p) = C_KM^T + D_KC_2X \Rightarrow C_K = (\mathbb{C}_0 - D_KC_2X)(M^\dagger)^T$$
$$\mathbb{D}(p) = D_K = \mathbb{D}_0$$

Chapter 21

Problem 21.1
The function $f(\Delta) = \Delta^2$ maps the half disk centered at the origin and with a radius ρ in the right half-plane into a full disk centered at the origin and with a radius ρ^2. Then, the function $\Delta_p = \frac{\rho^2+f}{\rho^2-f}$ maps the disk onto the whole right half-plane.
However, Δ cannot be expressed as a bilinear function of Δ_p. So, this mapping cannot be used in system design.

Problem 21.2
Straightforward calculation, omitted.

Problem 21.3
Refer to the reference:
Kang-Zhi Liu, Masao Ono and Xiaoli Li: Revisiting The Robust Performance Problem, Proc. of SICE2015, pp.650-653, Hangzhou (2015.07)