A Partial Parameterization of Nonlinear Output Feedback Controllers for Saturated Linear Systems ⋆

Kang-Zhi Liu [a] Daisuke Akasaka [b]

[a] *Dept. of Electrical and Electronic Engineering, Chiba University, 1-33 Yayoi-cho, Inage-ku, Chiba 263-8522, Japan*

[b] *The MathWorks GK, 4-15-1 Akasaka, Minato-ku, Tokyo 107-0052, Japan*

Abstract

This paper addresses the stabilization problem of linear systems subject to input saturation. The major purpose is to introduce nonlinearity into output feedback control laws so as to expand the design freedom for performance enhancement. To this end, a new approach is developed which involves a partial differential matrix inequality (PDMI). First, a global stability condition is derived based on a Lyapunov function and the passivity of the saturation function, which is characterized by a PDMI about the feedback law. Then, a class of stabilizing feedback laws is explicitly obtained by solving this PDMI analytically. The feedback laws are parameterized by a nonlinear function. Further, it is revealed that any linear observer can be used to realize the output feedback stabilization. Numerical examples, including the seek control of a hard disk drive, show that the introduced nonlinearity does contribute to the improvement of system performance. The application to integral control is also discussed.

Key words: Input saturation, nonlinear output feedback, parameterization of controllers, partial differential matrix inequality, integral control

1 Introduction

Linear systems with input saturation are commonly encountered in practice due to the inherent constraint on actuators. The dynamic performance of systems is severely constrained by the limitation of actuator. For example, in power systems the saturation of the magnetic excitor poses the greatest challenge to the transient stability of power generators[10]. The study of such systems has received great attention because of its importance in engineering practice. For this class of systems, it is difficult to achieve high performance only by linear feedback control due to the input saturation. In fact, it is well accepted that nonlinear feedback control generally outperforms linear feedback control for such systems.

Overview of existing methods

Historically, the control of saturated systems was first discussed in the context of integrator windup phenomenon. The well-known classic antiwindup method is to reduce the integrator gain through an inner loop with a dead-zone compensation when the actuator is saturated, refer to Fig. 1[5]. With the progress of robust control and nonlinear control, the recent trend is either to maximize the region of stability (for unstable plants) or to optimize the control performance. The tools mostly used are passivity, LPV and LMI.

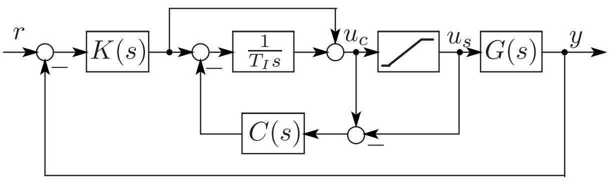


Fig. 1. Integrator antiwindup

Roughly speaking, the design philosophy falls into the following two categories.

(1) Design a low gain controller so as to avoid actuator saturation. The small-gain approach, such as [18], is along this line. A notable feature of this approach is that the potential of actuator is not effectively used. As a consequence, it is difficult to achieve high performance.

⋆ A part of this paper was presented at the 2008 IFAC World Congress.

Email addresses: kzliu@faculty.chiba-u.jp (Kang-Zhi Liu), daisuke.akasaka@mathworks.co.jp (Daisuke Akasaka).

(2) Actively make use of actuator and compensate the effect of saturation. The famous conditioning technique[5] and most of the recent literature are in this direction.

The key in saturation control is how to utilize the characteristic of the actuator saturation in system design. There are two major approaches that make use of different aspects of the saturation.

(1) Antiwindup approach: In this approach, a linear controller is designed without consideration of input saturation and $u_c - u_s$, the difference between the input command u_c and the output u_s of the actuator, is used to compensate the effect of actuator saturation[5]. Hence, the saturation is transformed into a dead-zone and treated as a sector nonlinearity. In recent years, the antiwindup compensator has been extended to include dynamics and, the feedback controller and antiwindup compensator are designed simultaneously[9, 4, 8, 12, 14, 20]. [16] is a good source on this topic which includes an extensive list of literature.
(2) Direct approach: The passivity and/or sector property of the saturation is used directly in control design, such as the Lyapunov approach of [7, 3].

From the viewpoint of control structure, most of the modern antiwindup methods use linear feedback controller and linear antiwindup compensator which are designed based on a quadratic Lyapunov function and may be regarded as extensions of Circle Criterion approach. LMI is used as a numerical optimization tool either to enlarge the region of stability or to achieve some induced $\mathcal{L}_2$ norm specification. Particularly, [4, Facts 2, 3 in Sec. V] succeeded in combining the induced $\mathcal{L}_2$ norm specification with passivity of dead-zone nonlinearity nicely via the well-known S-procedure.

Meanwhile, in the composite approach of [7, 3], some type of nonlinear controllers are proposed to enhance control performance. In the aspect of global stabilization of saturated systems, there have been many new developments. [15] proposed a nested design technique with a structure similar to the neural network. [17] proposed a scheduled H_∞-type control method by scheduling a parameter according to the size of the state. A design technique is proposed in [11] which schedules both the low-gain and the high-gain of the control law. [21] extended this gain scheduling approach and addressed the implementation issue. Moreover, [19] discussed the linear control for systems containing a double integrator, based on a Lur'e-Postnikov type Lyapunov function. A detailed comparison of many saturation control methods is conducted in [13] with respect to the double integrator.

Objective and contribution of this paper

Encouraged by the success of nonlinear control approaches mentioned above and our study on on-off switching systems[2], we try to pursue the nonlinear control of saturated systems from a different angle in this paper. The engineering motivation is that nonlinear control is able to achieve some wonderful performance, such as finite-time settling and nonlinear damping, which cannot be obtained by linear control. Even when the actuator saturation is faced, some of these nice properties may still be attained by making the best use of actuator. Another objective is to make full use of actuator power, rather than limiting it, so as to achieve higher performance. Our approach is quite different from others in that the structure of the state feedback controller is not prescribed. Instead, it is derived as the result of stabilization. In this way, we succeed in obtaining a partial parameterization of nonlinear state feedback controllers for saturated linear systems. Then, it is shown that any linear observer can be applied to the nonlinear state feedback to realize nonlinear output feedback control.

Concretely speaking, a Lur'e-Postnikov type Lyapunov function is used to derive the stability condition, which is characterized by a PDMI (partial differential matrix inequality) about the feedback law. Then a class of solutions is derived analytically which has a nonlinear integral kernel as free parameter. This leads to a partial parameterization of nonlinear controllers. The formula of solutions is explicit and easy to apply. This class is quite broad and, the controller is allowed to have arbitrarily high gain in particular. So there exists a high possibility that the freedom in the nonlinear controllers can be used to optimize the control performance.

Further, some examples are illustrated which show favorable system performance. Particularly, in the seek control design of hard disk drive (HDD), comparison with linear control shows the superior performance of the proposed control method.

This paper is organized as follows. After the statement of the problem, Sec. 3 provides the necessary mathematical results, Sec. 4 covers the designs of state feedback. The output feedback design is treated in Sec. 5 together with a brief design procedure. The case study on HDD is shown in Sec. 6. Further, the application to integral control is discussed in Sec. 7. For the sake of readability, all proofs are collected in the Appendix.

Notations For a matrix A, $A^\dagger$ denotes its Moore-Penrose inverse. $\ker A$ is the kernel space of A. Let Y_i be a square matrix, we use $\mathrm{diag}(Y_1, \cdots, Y_l)$ to denote the block diagonal matrix with diagonal block Y_i. Further, I_i denotes the ith dimensional identity matrix. $X_\perp$ is a row full rank matrix which has the largest rank among all matrices Y satisfying $YX = 0$. e_i is a vector whose ith entry is unit and the rest are all zeros. Also, $\mathrm{sgn}(\cdot)$ denotes the signum function.

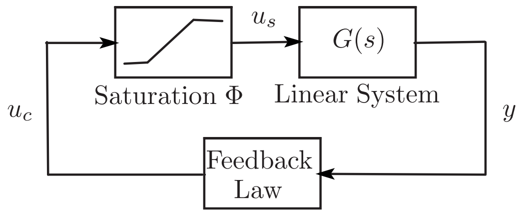


Fig. 2. Feedback control systems with saturation.

2 Problem statement

The feedback control system under consideration is illustrated Fig. 2, which consists of a multi-input multi-output linear system $G(s)$, input saturation Φ and a feedback law. Since this paper focuses on the stabilization issue, exogenous signal is omitted.

The state equation of the system G is described by

$$G : \begin{cases} \dot{x} = Ax + Bu_s, \ x \in \mathbb{R}^n, \ u_s \in \mathbb{R}^m \\ y = Cx, \qquad\qquad y \in \mathbb{R}^p. \end{cases} \quad (1)$$

The control input

$$u_s = \Phi(u_c) \quad (2)$$

is supplied through actuator with saturation. Here, u_c denotes the control command and, the saturation function $\Phi : \mathbb{R}^m \to \mathbb{R}^m$ is described as

$$\Phi(u_c) = [\phi_1(u_{c_1}) \ \cdots \ \phi_m(u_{c_m})]^T \quad (3)$$

where the ith element $\phi_i(u_{c_i})$ is any continuous function satisfying: (i) $\phi_i(u_{c_i}) = 0$ iff $u_{c_i} = 0$; (ii) $u_{c_i}\phi_i(u_{c_i}) \geq 0$ (passivity). A typical example is the ideal saturation:

$$\phi_i(u_{c_i}) = \begin{cases} u_{c_i}, & |u_{c_i}| \leq m_i \\ \mathrm{sgn}(u_{c_i})m_i, & |u_{c_i}| > m_i \end{cases}$$

where $m_i > 0$ denotes the maximum magnitude of the ith control input. Another example is the arctangent function

$$\phi_i(u_{c_i}) = \frac{2m_i}{\pi} \arctan\left(\frac{\pi}{2m_i} u_{c_i}\right).$$

It is worth noting that $\Phi(u_c) = 0$ iff $u_c = 0$.

Our goal is to design the input command u_c such that $u_s = \Phi(u_c)$ globally stabilizes the system (1). Both state and output feedback are considered. Since u_s is bounded due to the constraint of the actuator, not all linear systems are globally stabilizable. Thereafter, we make the following two assumptions on the system (1).

Assumption 1 (A, B, C) is controllable and observable.

Assumption 2 All eigenvalues of A are located in the closed left half plane. Further, the algebraic multiplicities of all Jordan blocks of zero eigenvalues are no greater than 2 and, all pure imaginary eigenvalues are simple.

Unstable systems, such as the double integrator, are included in this class of systems. Assumption 1 is made for the output stabilization. The first part of Assumption 2 is necessary for the global stabilizability with a bounded input. The algebraic multiplicity conditions in Assumption 2 are made to ensure (i) the existence of a nontrivial matrix P satisfying

$$P \geq 0, \quad A^T P + PA \leq 0, \quad \ker A \subset \ker P; \quad (4)$$

(ii) the radial unboundedness of a certain Lyapunov function [1]. All these properties play important roles in the feedback design. The characterization of such P is given in the next section.

3 Mathematical preliminaries

3.1 Solution of nonstrict Lyapunov inequality

Owing to Assumption 2, we can assume without loss of generality that

$$A = \mathrm{diag}(A_o, \ A_\omega, \ A_s). \quad (5)$$

The dimensions of square matrices A_o, A_ω, A_s are denoted by n_o, n_ω, n_s respectively. All eigenvalues of A_o are zeros and A_o has the form of

$$A_o = \mathrm{diag}(J_1, \cdots, J_q) \quad (6)$$

in which the Jordan block $J_i \in \mathbb{R}^{k_i \times k_i}$ is 0 for $k_i = 1$ and, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ for $k_i = 2$. All eigenvalues of A_ω are on the imaginary axis except the origin and A_ω is given by

$$A_\omega = \mathrm{diag}(\begin{bmatrix} 0 & \omega_1 I_{l_1} \\ -\omega_1 I_{l_1} & 0 \end{bmatrix}, \cdots, \begin{bmatrix} 0 & \omega_t I_{l_t} \\ -\omega_t I_{l_t} & 0 \end{bmatrix}) \quad (7)$$

where $\omega_i > 0$ for $i = 1, \cdots, t$. Further, all eigenvalues of A_s have negative real parts.

The input matrix B and the state vector x are decomposed as

$$B = [B_o^T \ B_\omega^T \ B_s^T]^T, \quad x = [x_o^T \ x_\omega^T \ x_s^T]^T \quad (8)$$

[1] This point will be further clarified after Theorem 2.

in accordance with (6) hereafter. Moreover, it is assumed that the pair (A_o, B_o) is in the controllable canonical form. That is, the input matrix B_{oi} corresponding to the Jordan block $J_i \in \mathbb{R}^{k_i \times k_i}$ of (8) has a form of

$$\begin{cases} B_{oi} = \begin{bmatrix} 0 & \cdots & 0 & 1 & * & \cdots & * \end{bmatrix}, \ k_i = 1 \\ B_{oi} = \begin{bmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & * & \cdots & * \end{bmatrix}, \ k_i = 2 \end{cases} \tag{9}$$

where the entry 1 appears in the ith column and $*$ stands for a possibly nonzero number.

Since both A_ω and A_s are nonsingular, it is easy to see that (note that $A_o^\dagger = A_o^T$ in the present coordinate)

$$A_\perp = [A_{o\perp} \ 0 \ 0], \quad A^\dagger = \mathrm{diag}(A_o^T, \ A_\omega^{-1}, \ A_s^{-1}). \tag{10}$$

Solutions of the nonstrict inequality (4) can be constructed as follows.

Lemma 1 *Subject to Assumption 2, solution of (4) exists and is given by*

$$P = \mathrm{diag}(P_o, P_\omega, P_s) \tag{11}$$

where $P_o \in \mathbb{R}^{n_o \times n_o}$, $P_\omega \in \mathbb{R}^{n_\omega \times n_\omega}$ and $P_s \in \mathbb{R}^{n_s \times n_s}$ are characterized as follows.

(1) P_o is positive semi-definite and equal to

$$P_o = \mathrm{diag}(R_1, \cdots, R_r) \tag{12}$$

where $R_i \in \mathbb{R}^{k_i \times k_i}$ is 0 for $k_i = 1$ and, $R_i = \mathrm{diag}(0, r_i)$ $(r_i > 0)$ for $k_i = 2$.

(2) P_ω is positive definite and has the form:

$$P_\omega = \mathrm{diag}(\begin{bmatrix} X_1 & Y_1 \\ Y_1^T & X_1 \end{bmatrix}, \cdots, \begin{bmatrix} X_t & Y_t \\ Y_t^T & X_t \end{bmatrix}) \tag{13}$$

where $X_i \in \mathbb{R}^{l_i \times l_i}$ is any positive definite matrix and $Y_i \in \mathbb{R}^{l_i \times l_i}$ is any skew-symmetric matrix.

(3) P_s is positive definite and satisfies

$$A_s^T P_s + P_s A_s + Q_s = 0, \quad Q_s > 0. \tag{14}$$

(4) In addition, the matrix P given in (11) satisfies the following Lyapunov equation

$$A^T P + PA + Q = 0, \quad Q = \mathrm{diag}(0, \ 0, \ Q_s). \tag{15}$$

(5) Further, there holds

$$B^T [PA^\dagger + (A^\dagger)^T P + (A^\dagger)^T Q A^\dagger] B = 0. \tag{16}$$

Most of the conclusions follow from arguments similar to [19] and the proof is omitted. Only the statement (5) is shown here. It is easy to see that $PA^\dagger + (A^\dagger)^T P + (A^\dagger)^T Q A^\dagger = \mathrm{diag}(P_o A_o^T + A_o P_o, 0, 0)$. So, we have $B^T [PA^\dagger + (A^\dagger)^T P + (A^\dagger)^T Q A^\dagger] B = B_o^T [P_o A_o^T + A_o P_o] B_o$. Then, the conclusion follows from direct calculation based on the structure of (A_o, B_o).

The following factorization of matrix Q will be used throughout the paper

$$Q = C_Q^T C_Q, \quad C_Q = [0 \ 0 \ Q_s^{1/2}] \in \mathbb{R}^{n_s \times n}. \tag{17}$$

3.2 Analytical solution to PDMI

In this subsection, a key to the synthesis of stabilizing nonlinear feedback law is presented. The stability condition to be shown after this section is basically characterized by a PDMI about the feedback law, which is derived from a Lur'e-Postnikov type Lyapunov function (refer to Appendix B for the detail). To build the stabilizing feedback law, this PDMI must be solved.

Specifically, the main issue is to find an analytical solution $S(x) \in \mathbb{R}^m$ for the PDMI below

$$\begin{bmatrix} A^T P + PA & PB + A^T (\frac{\partial S}{\partial x})^T \Lambda \\ B^T P + \Lambda \frac{\partial S}{\partial x} A & \Lambda \frac{\partial S}{\partial x} B + B^T (\frac{\partial S}{\partial x})^T \Lambda \end{bmatrix} \le 0 \tag{18}$$

where $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_m) > 0$. The vector function $S(x)$ is described as

$$S(x) = [s_1(x) \ \cdots \ s_m(x)]^T, \quad S(0) = 0 \tag{19}$$

$S(0) = 0$ is imposed in order to ensure that the origin is an equilibrium. The partial derivative $\frac{\partial S}{\partial x}$ is

$$\frac{\partial S}{\partial x} = \Big[\Big(\frac{\partial s_1}{\partial x}\Big)^T \ \cdots \ \Big(\frac{\partial s_m}{\partial x}\Big)^T \Big]^T. \tag{20}$$

The function $S(x)$ in (18) is the key component in the feedback laws. In fact, the state feedback law is given by $u_c = S(x)$. On the other hand, the output feedback control is implemented via $u_c = S(\hat{x})$ in which $\hat{x}$ is the state estimate generated from some state observer.

In general, it is hard to obtain an analytical solution for partial differential equations in nonlinear optimal control problems, such as Pontryagin's maximum principle and Hamilton-Jacobi-Bellman equation. These problems usually have to be solved numerically. Fortunately, for the special problem (18) analytical solution can be found. The detail is given in the following theorem.

It is easy to know that the rank of $A_\perp$ is equal to q, the number of Jordan blocks in A_o.

Theorem 1 (Appendix A) *Define a function* $\alpha_j(x)$ *as*

$$\alpha_j(x) = e_j^T A_\perp x = e_j^T A_{o\perp} x_o, \ j = 1, \dots, q. \quad (21)$$

Suppose that $G(x) = (g_{ij}(\alpha_j(x))) \in \mathbb{R}^{q\times q}$ *is a piecewise continuous matrix function,* $F \in \mathbb{R}^{q\times n_s}$ *is a constant matrix and define a matrix function* $\Pi(x)$ *as*

$$\Pi(x) = G(x) + G^T(x) - FF^T. \quad (22)$$

The PDMI (18) has the following solutions.

(1) When A *is singular and* $\Pi(x) \geq 0$ *for all* x*, then*

$$S(x) = -\Lambda^{-1}B^T[P + (A^\dagger)^T Q + (A_\perp)^T F C_Q]A^\dagger x - \Lambda^{-1}(A_\perp B)^T \sum_{j=1}^{q} \int_0^{\alpha_j(x)} \begin{bmatrix} g_{1j}(w) \\ \vdots \\ g_{qj}(w) \end{bmatrix} dw. \quad (23)$$

(2) When A *is nonsingular, then the solution is unique and equal to*

$$S(x) = \Lambda^{-1}(A^{-1}B)^T Px. \quad (24)$$

Theorem 1 provides a class of solutions for the PDMI (18). The free parameters in $S(x)$ and their roles in feedback control are summarized below.

(1) It is always possible to design the free parameters $G(x)$ and F such that the condition $\Pi(x) \geq 0$ is satisfied.
(2) The major parameter in $S(x)$ is the nonlinear integral kernel $G(x)$. This function can be freely chosen so long as the constraint $\Pi(x) \geq 0$ is satisfied. This means that $G(x)$ is not bounded above. The nonlinearity coming from the integral term of $G(x)$ depends only on the states of integrators since $A_\perp x = A_{o\perp} x_o$.
(3) The linear term concerning F (i.e., $(A_\perp B)^T F C_Q A^\dagger x$) only depends on the states of the stable dynamics because $C_Q A^\dagger x = Q_s^{1/2} A_s^{-1} x_s$.
(4) There is a linear term $P_\omega A_\omega^{-1} x_\omega$ whose gain can be tuned by the arbitrary parameters (X_i, Y_i).
(5) One more parameter is the matrix Q_s in the Lyapunov equation which changes P_s and affects the linear term $P_s A_s^{-1} x_s$.
(6) The first part of $S(x)$ has a linear term $P_o A_o^\dagger x_o$ which contains those integrator states that are not in $A_{o\perp} x_o$. The gain of this term can be tuned by the free parameter $r_i > 0$ contained in P_o.
(7) Finally, the gain of $S(x)$ increases as Λ decreases.

In a word, all states are contained in $S(x)$ and the gain of each state can be freely tuned.

4 State feedback design

We start with stabilizing the plant by a general state feedback law

$$u_c = S(x). \quad (25)$$

Then, the closed loop system is described as

$$\dot{x} = Ax + B\Phi(S(x)). \quad (26)$$

It is only assumed that $S(x)$ is a piecewise continuously differentiable vector function with $S(0) = 0$.

The following theorem is the main result which parameterizes a class of stabilizing state feedback laws.

Theorem 2 (Appendix B) *Suppose that Assumptions 1 and 2 hold and the matrix* $\Pi(x)$ *is positive definite. Then the feedback law* $S(x)$ *parameterized in Theorem 1 guarantees that the state* $x(t)$ *globally converges to the largest invariant set* $\mathcal{V}$ *contained in the set:*

$$\Omega = \{x : C_Q(x + A^\dagger B\Phi) = 0, \ A_\perp B\Phi = 0\}. \quad (27)$$

Further, the state $x(t)$ *converges to the origin asymptotically when* $q = m$ *and* $G(x)$ *is a lower triangular matrix.*

This theorem is derived based on the Lur'e-Postnikov type Lyapunov function:

$$V(x) = x^T Px + 2\sum_{i=1}^{m} \lambda_i \int_0^{s_i(x)} \phi_i(s)ds \ (\lambda_i > 0). \quad (28)$$

The stability condition $\dot{V}(x) \leq 0$ leads to the PDMI (18). Then Theorem 1 provides the solutions.

Basically, in Lyapunov approach the radial unboundedness property of the Lyapunov function is required in order to guarantee the global stability. The algebraic multiplicity conditions made in Assumption 2 are necessary for the radial unboundedness of the Lyapunov function above. To illustrate it, consider the case where $A = \begin{bmatrix} 0 & I_2 \\ 0 & 0 \end{bmatrix}$. Then, the solution of the Lyapunov equation (15) turns out to be $P = \mathrm{diag}(0, 0, r)$ [19]. Let the corresponding states be $x = [x_1, x_2, x_3]^T$, then $x^T Px$ contains only x_3. Moreover, since $A_\perp = [0\ 0\ 1]$ and $A^\dagger = A^T$, it can be shown that $S(x)$ does not contain x_1. Therefore, $V(x)$ does not have the state x_1 and is not radially unbounded. This also happens when any pure imaginary eigenvalue has an algebraic multiplicity greater than one.

Remark 1 *As shown in Appendix B, the derivative of* $V(x)$ *is equal to (note that* $(\#)^T X = X^T X$*)*

$$\dot{V}(x) = -[\#]^T[C_Q x + (C_Q A^\dagger + F^T A_\perp)B\Phi] - (A_\perp B\Phi)^T \cdot \Pi(x) \cdot A_\perp B\Phi. \quad (29)$$

Roughly speaking, the second term in $\dot{V}(x)$ is minimized when $u_s = \Phi(S(x))$ saturates. Since $V(x)$ is a positive definite function, it can be expected that the state $x(t)$ converges faster if $\dot{V}(x)$ gets smaller. Hence, usually it is better to use high gain in the feedback law $S(x)$. It is clear that large $G(x)$ yields large $\Pi(x)$, which also contributes to the convergence.

Meanwhile, nonzero F reduces $\Pi(x)$ and does not contribute to the convergence of $V(x)$. So, it is better to set F as zero from this viewpoint. Of course, nonzero F may be introduced if additional feedback of x_s is necessary.

Next, let us investigate the property of the set Ω for the general case. It is noted that $\Omega = \{x : Q_s A_s^{-1}(A_s x_s + B_s\Phi) = 0,\ A_{o\perp}B_o\Phi = 0\}$. The first equation implies that $\dot{x}_s = 0$, i.e., x_s is a constant vector. Meanwhile, the second equation means that $B_o\Phi = 0$ so that x_o obeys $\dot{x}_o = A_o x_o$. The boundedness of x_o follows from that of $V(x)$, so x_o must be a constant vector. In particular, for a double integrator its velocity must be zero.

Further, $q = m$ implies that each input channel has at least one integrator. Since $A_\perp[B\ AB\ A^2B\ \cdots] = [A_\perp B\ 0\ 0\ \cdots]$, $A_\perp B$ always has full row rank and is nonsingular when $q = m$. In this case, $\dot{V}(x)$ gets even smaller. This property, together with the asymptotic convergence, implies that the proposed approach is particularly effective for systems with integrator.

Example 1 *Consider an integrator $1/s$ whose realization is $A = 0, B = C = 1$. It is easy to know that $\alpha(x) = x$ and $\Pi(x) = 2g(x)$ ($f = 0$). When we choose $g(w) = |w|^{-1/2}$, then $S(x) = -\frac{2}{\lambda}\mathrm{sgn}(x)|x|^{1/2}$. So when the actuator is saturated (the largest magnitude is 1), the state equation is $\dot{x} = \mathrm{sgn}(S) = -\mathrm{sgn}(x)$ (Bang-bang control) and $x(t)$ converges to the interval $[-\lambda^2/4, \lambda^2/4]$ in finite time. After that, $x(t)$ obeys the dynamics $\dot{x} = -\frac{2}{\lambda}\mathrm{sgn}(x)|x|^{1/2}$ whose solution is $\sqrt{|x(t)|} = \sqrt{|x(0)|} - t/\lambda$. Again $x(t)$ converges to zero in finite time. This performance cannot be achieved by linear feedback. To reduce the settling time, it suffices to decrease λ. In this nonlinear control the full power of actuator is actively used.*

Next, we have a brief discussion on the relation between the tuning gain λ_i and the state convergence. For this purpose, a function

$$s_i^\circ(x) = \lambda_i s_i(x), \quad i = 1, \cdots, m \tag{30}$$

is defined which is independent of λ_i. In the saturated domain, $|s_i(x)| \geq m_i \Leftrightarrow |s_i^\circ(x)| \geq \lambda_i m_i$ ($i = 1, \cdots, m$). The convergence rate of $V(x)$ is very big and

$$V(x) = x^T P x + 2\sum_{i=1}^{m} |s_i^\circ(x)|. \tag{31}$$

So both $x^T P x$ and $|s_i^\circ(x)|$ converge quickly. In particular, $s_i^\circ(x)$ converges to $|s_i^\circ(x)| < \lambda_i m_i$. By the radial unboundedness of $V(x)$, all states are contained in it. Therefore, the state converges to a compact set and this set can be made smaller by reducing λ_i. This means that a rule of thumb in the selection of λ_i is to reduce it until the input chatters.

5 Output feedback and design procedure

In this section, the output feedback design is treated. The approach is quite standard: we use an observer to estimate the state so as to realize the state feedback law designed in the previous section. The available information are the output $y = Cx \in \mathbb{R}^p$ and the plant input $u_s = \Phi(u_c)$.

The following observer

$$\begin{aligned} \dot{\xi} &= T\xi + Hu_s + Ly \\ \hat{x} &= J\xi + Ny \end{aligned} \tag{32}$$

is constructed where $\xi \in \mathbb{R}^{n_\xi}$ is the partial state variable to be estimated and $\hat{x}$ is the observed state vector. It is well-known that for this observer to be able to recover the state x, its coefficient matrices must satisfy the following two conditions:

(1) T is a Hurwitz matrix.
(2) There exists a $n_\xi \times n$ matrix U satisfying

$$UA - TU = LC,\ UB - H = 0,\ NC + JU = I. \tag{33}$$

The existence of such matrix U is guaranteed by the observability of (C, A).

The output feedback is implemented by replacing the unknown state x by its estimate $\hat{x}$ in the state feedback law $S(\cdot)$, i.e. constructing the input command by

$$u_c = S(\hat{x}). \tag{34}$$

The resulting closed loop system is then described by

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} Ax + B\Phi(S(x - Je)) \\ Te \end{bmatrix}. \tag{35}$$

in which $e = Ux - \xi$ denotes the estimation error.

The global stability condition for the output feedback is as follows. A brief proof can be found in [1].

Theorem 3 *Suppose that all conditions of Theorem 2 are satisfied. If the function $S(x)$ given in Theorem 1 is* *selected as such that it satisfies*

$$\|\frac{\partial s_i}{\partial x}(x)\|\|x\| \leq \delta_i |s_i(x)| \quad \forall \|x\| \geq M \tag{36}$$

for some positive constants M, δ_i $(i = 1, \ldots, m)$, then the state vector $(x(t), e(t))$ of the closed loop system (35) globally converges to $\mathcal{V} \times \{0\}$.

Remark 2 *The condition (36) in Theorem 3 requires the polynomial growth of $s_i(x)$ for $\|x\| \geq M$. If $s_i(x)$ is a polynomial-like function (refer to Example 1 and Sec. 6 for examples), it satisfies the condition (36).*

Based on the preceding results, the design flow of the global stabilizer can be briefly summarized as follows.

1) Choose a matrix $Q_s > 0$ and solve the Lyapunov equation $A_s^T P_s + P_s A_s + Q_s = 0$. Then compute the matrix $P \geq 0$ following Lemma 1 of Section 3.1.
2) Pick matrices $(G(x), F)$ as such that the matrix inequality $\Pi(x) \geq 0$ is satisfied and compute $S(x)$ according to (23).
3) Design the observer (32) using some method, such as pole placement.
4) Reducing λ_i $(i = 1, \cdots, m)$ until satisfactory response is obtained.

The design parameters are the constant matrix Q_s, function $(G(x), F)$, as well as the tuning gain λ_i.

6 A case study

In this section, we illustrate an output feedback design example about the seek control of an HDD. The plant is given by the fourth order plant below[6]

$$G(s) = K_p\left(\frac{1}{s^2} - \frac{\kappa\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n}\right) \tag{37}$$

which consists of a double integrator and a second order vibration mode. Here $K_p = 3.74 \times 10^9$, $\zeta = 0.31$, $\omega_n = 4100$[Hz] and $\kappa = 0.7/\omega_n^2$. The control objective of this example is to make the plant output y track the constant reference signal r. To this end, the input y to the observer (32) is replaced by $y - r$.

A minimal state-space realization of (37) is obtained as

$$A = \begin{bmatrix} A_o & 0 \\ 0 & A_s \end{bmatrix}, \quad B = \begin{bmatrix} b_o \\ b_s \end{bmatrix}, \quad C = \begin{bmatrix} c_o & c_s \end{bmatrix} \tag{38}$$

where

$$A_o = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; A_s = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix}, \; b_o = b_s = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$c_o = K_p \begin{bmatrix} 1 & 0 \end{bmatrix}, \; c_s = K_p \begin{bmatrix} -\kappa\omega_n^2 & 0 \end{bmatrix}$$

To design the output feedback law, we follow the design procedure summarized in Sec. 5.

Step 1) The solution $P = \mathrm{diag}(P_o, P_s)$ of Lyapunov equation (15) is computed as

$$P_o = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \; P_s = \begin{bmatrix} 1.906 \times 10^7 & 2082.1 \\ 2082.1 & 0.819 \end{bmatrix}$$

with respect to matrix $Q_s = \mathrm{diag}(7 \times 10^{10}, 1)$.

Step 2) Decompose x_o as $x_o = [x_{o1} \; x_{o2}]^T$. We have $A_\perp x = x_{o2}$. The constraint $\Pi(x) > 0$ reduces to $g(x_{o2}) > 0$ when $f = 0$. Here, $g(\cdot)$ is selected as

$$g(w) = \frac{g_1}{2}|w| + g_2, \quad g_1, g_2 > 0. \tag{39}$$

The nonlinear output feedback law given in (23) is then computed as

$$S(x) = -\tfrac{1}{\lambda}\left[p_{22}x_{o1} - (A_s^{-1}b_s)^T P_s x_s + g_1|x_{o2}|x_{o2} + g_2 x_{o2}\right] \tag{40}$$

where p_{22} is the $(2,2)$-th entry of P_o. This $S(x)$ is a polynomial growth function. Parameters (λ, g_1, g_2) are tuned as $(10^{-11}, 1/1.75, 5 \times 10^{-4})$ respectively. Basically, this is a high-gain controller.

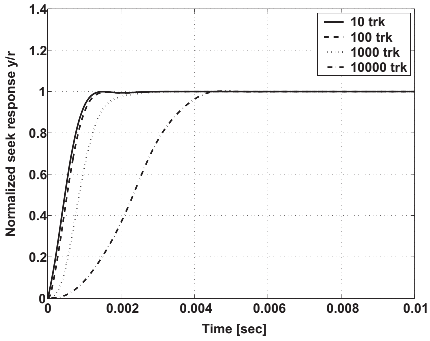
Step 3) A full order observer is used, i.e. $U = I$, $T = A - LC$, $N = 0$ and $J = I$. The observer gain L is designed as $L = [1.1553 \times 10^{-11},\ 1.7306 \times 10^{-10},\ 8.8695 \times 10^{-7},\ 0.0042]^T$.

In simulation, the ideal saturation function is used, with $|u| \leq 0.5$[A]. Fig. 3 shows the responses of the plant output and the saturated input for constant reference values $r = 10, 100, 1000, 10000$[trk]. Note that in Fig. 3(a) the plant output value is normalized so that it converges to 1 for each reference. As seen in Fig. 3(a), the plant output converges to each reference value quickly without integrator windup.

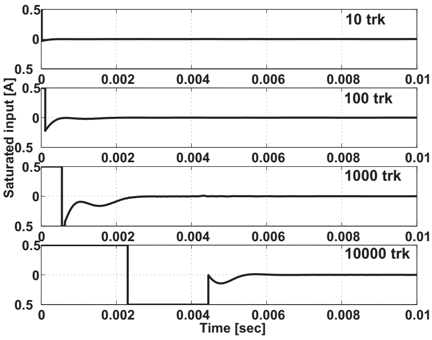
For comparison, Fig. 4 shows the result of a linear output feedback law which consists of the state feedback law

$$S(x) = -10^{11}(x_1 + 0.4 \times 10^{-3}x_2 + 10^{-4}x_3 - 1.5 \times 10^{-4}x_4) \tag{41}$$

together with the same observer. The feedback gain of (41) is tuned by trial and error such that a fast convergence rate is achieved for $r = 10, 100$[trk]. However, for



(a) Plant output



(b) Saturated input

Fig. 3. Responses of saturated system with nonlinear control. From above, each input curve corresponds to r = 10, 100, 1000, 10000[trk] respectively.

larger reference values windup occurs in the linear control while the nonlinear control demonstrates a fast convergence without large overshoot, as seen in Fig. 3(a) and Fig. 4(a).

This numerical example exhibits that the introduced nonlinear control outperforms linear control for a broad range of reference values. Also, significant difference between control actions can be observed from Fig. 3(b) and Fig. 4(b). Although both start with full acceleration, then turn to full deceleration, the timings are quite different for large references.

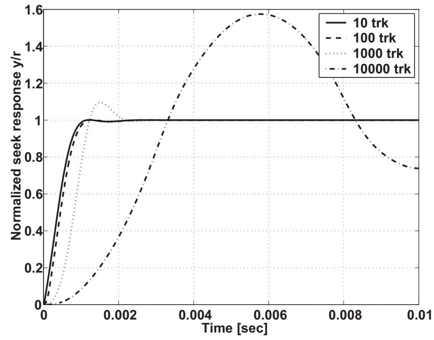
7 Application to integral control

It is well-known that the integrator control is necessary in servo-mechanism problems. The proposed approach of this paper can be applied to avoid integrator windup. Consider the integral control system in Fig. 5. Here the plant $P(s)$ has a state equation

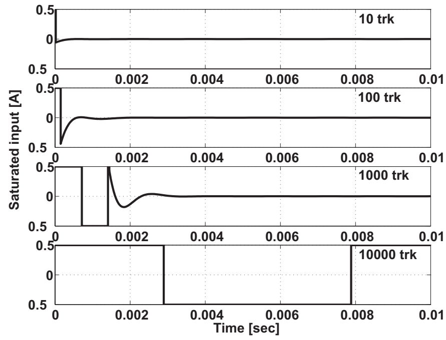
$$\dot{x}_P = A_P x_P + B_P u_s, \quad y = C_P x_P. \tag{42}$$

Obviously, the following two conditions are necessary for the asymptotic tracking of step reference input.

Condition 1 The number of input u_s must not be less than that of output y.



(a) Plant output



(b) Saturated input

Fig. 4. Responses of saturated system with linear control. From above, each input curve corresponds to r = 10, 100, 1000, 10000[trk] respectively.

Condition 2 $P(s)$ does not have any zero at $s = 0$, i.e. $\begin{bmatrix} A_P & B_P \\ C_P & 0 \end{bmatrix}$ has full row rank.

Let the state vector of the integrator $\frac{1}{s}I$ be x_I. Then the enlarged plant $G(s)$ has the following state realization:

$$\begin{bmatrix} \dot{x}_I \\ \dot{x}_P \end{bmatrix} = A \begin{bmatrix} x_I \\ x_P \end{bmatrix} + B u_s, \quad y_G = C \begin{bmatrix} x_I \\ x_P \end{bmatrix} \tag{43}$$

$$A = \begin{bmatrix} 0 & C_P \\ 0 & A_P \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ B_P \end{bmatrix}, \; C = \begin{bmatrix} I & 0 \\ 0 & C_P \end{bmatrix}$$

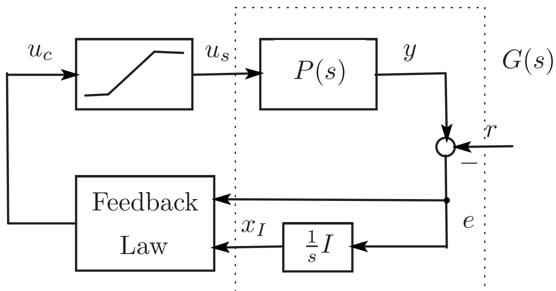


Fig. 5. Integral control system

in which the reference signal r has been omitted since we focus on the stabilization issue. Let $[X \;\; Y] = \begin{bmatrix} A_P \\ C_P \end{bmatrix}_\perp$. Then $[X \;\; Y]\begin{bmatrix} A_P & B_P \\ C_P & 0 \end{bmatrix} = [0 \;\; XB_P] \neq 0$. implies $XB_P \neq 0$. So

$$A_\perp = \begin{bmatrix} Y & X \\ 0 & A_{P\perp} \end{bmatrix} \Rightarrow A_\perp B = \begin{bmatrix} X \\ A_{P\perp} \end{bmatrix} B_P \neq 0$$

hold. Thus, nonlinear terms can be introduced to the feedback law.

Two examples about SISO systems are shown below.

Example 2 *The first example is the case where the plant does not have any integrator. When the state vector of the enlarged plant $G(s)$ is chosen as $x = [x_I - c_p A_P^{-1} x_P \;\; x_P^T]^T$, the coefficient matrices of the realization becomes*

$$A = \begin{bmatrix} 0 & 0 \\ 0 & A_P \end{bmatrix},\; B = \begin{bmatrix} -c_p A_P^{-1} b_P, & b_P \end{bmatrix},\; C = \begin{bmatrix} 1 & c_p A_P^{-1} \\ 0 & c_P \end{bmatrix}.$$

Then $A_\perp = [1 \;\; 0]$, $A_\perp B = -c_p A_P^{-1} b_P \neq 0$ so that nonlinear term about $\alpha(x) = A_\perp x = x_I - c_p A_P^{-1} x_P$ is introduced into the feedback law $S(x)$.

The second example is the case where the plant has one integrator. Suppose that the state vector of the plant $P(s)$ is $x_P = [x_o \;\; x_1^T]^T$ and the corresponding state space realization is

$$A_P = \begin{bmatrix} 0 & 0 \\ 0 & A_1 \end{bmatrix},\; b_P = \begin{bmatrix} b_o \\ b_1 \end{bmatrix},\; c_P = [1 \;\; c_1].$$

Then, it is easy to know that by selecting the state vector of the enlarged plant $G(s)$ as $x = [x_I - c_1 A_1^{-1} x_1 \;\; x_o \;\; x_1^T]^T$, the coefficient matrices of the realization simplifies to

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & A_1 \end{bmatrix},\; B = \begin{bmatrix} -c_1 A_1^{-1} b_1 \\ b_o \\ b_1 \end{bmatrix},\; C = \begin{bmatrix} 1 & 0 & c_1 A_1^{-1} \\ 0 & 1 & c_1 \end{bmatrix}$$

and $A_\perp = [0 \;\; 1 \;\; 0]$, $A_\perp B = b_o \neq 0$. Then, nonlinear term about $\alpha(x) = A_\perp x = x_o$ is introduced.

8 Concluding remarks

In this paper, the nonlinear feedback stabilization problem has been studied for linear systems with input saturation. A completely new approach is proposed which is based on the analytical solution of a partial differential matrix inequality. The main contributions of the paper are two folds:

(1) a new approach based on the analytic solution of partial differential matrix inequality,
(2) a partial parameterization of stabilizing nonlinear output feedback laws for systems with integrator.

Numerical examples indicate that the nonlinear control law does have advantage in resolving integrator windup if the nonlinear function in the feedback law is suitably selected.

However, many issues remain open.

(1) In the HDD case study, the nonlinear integral kernel is found via phase plane analysis about the rigid body model (double integrator). It is desirable to establish some systematic methods for the determination of the nonlinear function in the control law so as to optimize the performance. Variational approach might be helpful in this direction.
(2) In Theorem 2 only the passivity of saturation function was used. If the sector property of the actuator is used, we will end up with a PDMI which cannot be solved via integration. At present, it is not clear how to solve such PDMI. Further, the bound of actuator has not been used. If all these features of saturation can be effectively utilized in the control design, even better performance may be expected.
(3) The robustness issue of saturated control is also important. This problem is under study.

Acknowledgements

The authors would like to thank the Associate Editor and the anonymous reviewers for their constructive comments and suggestions that help improving the quality of this paper.

References

[1] Akasaka, D., Liu, K. Z. (2008). Nonlinear output feedback control of linear systems with input saturation, *Proc. 17th IFAC World Congress*, 15154-15159.

[2] Akasaka, D., Liu, K. Z. (2007). Nonlinear feedback control of a class of on-off control systems: an approach based on analytical solution of PDEs, *Proc. 46th IEEE CDC*, 86-91.

[3] Chen, B. M., Lee, T. H., Peng, K., Venkataramanan, V. (2003). Composite nonlinear feedback control for linear systems with input saturation: theory and an application. *IEEE Trans. AC*, 48(3), 427-439.

[4] Grimm, G., Hatfield, J., Postlethwaite, I., Teel, A. R., Turner, M. C., Zaccarian, L. (2003). Anti-windup for stable linear systems with input saturation: An LMI-based synthesis. *IEEE Trans. AC*, 48(9), 1509-1525.

[5] Hanus, R., Kinnaert, M., Henrotte, J. L. (1987). Conditioning technique, a general anti-windup and bumpless transfer method. *Automatica*, 23(6), 729-739.

[6] Hirata, M., Hasegawa, Y. (2007). High bandwidth design of track-following control system of hard disk drive using H_∞ control theory. *Proc. IEEE CCA*, 114-117.

[7] Hu, T., Lin, Z. (2003). Composite quadratic Lyapunov functions for constrained control systems. *IEEE Trans. AC*, 48(3), 440-450.

[8] Hu, T., Teel, A. R., Zaccarian, L. (2009). Anti-windup synthesis for linear systems with input saturation: achievable regional, nonlinear performance. *Automatica*, 44, 512-519.

[9] Kothare, M. V., Campo, P., Morari, M., Nett, C. (1994). A unified framework for the study of anti-windup designs. *Automatica*, 30(12), 1869-1883.

[10] Kundur, P. (1994). *Power System Stability and Control*. McGraw-Hill.

[11] Lin, Z. (1998). Global control of linear systems with saturating actuators, *Automatica*, 34(7), 897-905.

[12] Mulder, E. F., Kothare, M. V., Morari, M. (2001). Multivariable anti-windup controller synthesis using linear matrix inequalities. *Automatica*, 37(9), 1407-1416.

[13] Rao, V. G. and Bernstein, D. S.(2001). Naive control of the double integrator, *IEEE Control Systems*, 21(5), 86-97.

[14] Saeki, M., Wada, N. (2002). Synthesis of a static anti-windup compensator via linear matrix inequalities. *Int. J. Robust Nonlinear Control*, 12, 927-953.

[15] Sussmann, H. J., Sontag, E. D. and Yang, Y.(1994). A general results on the stabilization of linear systems using bounded controls. *IEEE Trans. AC*, 39(12), 2411-2425

[16] Tarbouriech, S., Turner, M. (2009). Anti-windup design: an overview of some recent advances and open problems. *IET Control Theory Appl.*, 3(1), 1-19.

[17] Teel, A. R. (1995). Linear systems with input nonlinearities: global stabilization by scheduling a family of H_∞-type controllers. *Int. J. Robust and Nonlinear Control*, 5, 399-411.

[18] Teel, A. R. (1996). A nonlinear small gain theorem for the analysis of control systems with saturation. *IEEE Trans. AC*, 41(9), 1256-1270.

[19] Tyan, F. and Berstein, D. S.(1999). Global stabilization of systems containing a doule integrator using a saturated linear controller. *Int. J. Robust Nonlinear Control*, 9, 1143-1156.

[20] Wang, J. Z., Chan, C. W., Zhang, J. F. (2004). Global stability of systems with amplitude and rate saturation compensation. *Int. J. Robust Nonlinear Control*, 17(12), 155-170.

[21] Zhou, B., Lin, Z., Duan, G. (2010). Robust global stabilization of linear systems with input saturation via gain scheduling, *Int. J. Robust Nonlinear Control*, 20(4), 424-447.

A Proof of Theorem 1

The idea of solution is as follows. First, (18) is reduced to a PDME (partial differential matrix equation) and a matrix inequality condition. Then, the PDME is solved analytically and the matrix inequality puts a constraint on the free parameters of the solution.

This idea is concretely formalized by the following proposition, which provides an equivalent condition for (18).

Proposition 1 *The following two statements are equivalent:*

(1) The PDMI (18) holds.
(2) There exists an $m \times n_s$ matrix $K(x)$ satisfying

$$B^T P + \Lambda \frac{\partial S}{\partial x} A = -K(x) C_Q \tag{A.1}$$

$$\Lambda \frac{\partial S}{\partial x} B + B^T (\frac{\partial S}{\partial x})^T \Lambda + K(x) K^T(x) \leq 0. \tag{A.2}$$

PROOF. Since $A^T P + PA = -Q = -C_Q^T C_Q \leq 0$, (18) is equivalent to

$$\ker C_Q = \ker Q \subset \ker \big(B^T P + \Lambda \frac{\partial S}{\partial x} A\big) \tag{A.3}$$

and

$$\begin{aligned} &\Lambda \frac{\partial S}{\partial x} B + B^T (\frac{\partial S}{\partial x})^T \Lambda \\ &+\big(B^T P + \Lambda \frac{\partial S}{\partial x} A\big) Q^\dagger \big(B^T P + \Lambda \frac{\partial S}{\partial x} A\big)^T \leq 0 \end{aligned} \tag{A.4}$$

via a Schur complement argument. For fixed x, (A.1) can be regarded as a linear matrix equation about $K(x)$. The existence condition of its solution is exactly (A.3). Finally, substitution of (A.1) into (A.4) yields

$$\Lambda \frac{\partial S}{\partial x} B + B^T (\frac{\partial S}{\partial x})^T \Lambda + K(x) C_Q Q^\dagger C_Q^T K^T(x) \leq 0. \tag{A.5}$$

Hence (A.2) is obtained from $C_Q Q^\dagger C_Q^T = I$. □

The first step in solving the PDME (A.1) is to compute $\frac{\partial S}{\partial x}$ from it. The result is summarized below.

Lemma 2 *(A.1) has a solution* $\frac{\partial S}{\partial x}$ *iff*

$$\ker A \subset \ker P. \tag{A.6}$$

All solutions are parameterized by

$$\frac{\partial S}{\partial x} = -\Lambda^{-1}[B^T P A^\dagger + K(x) C_Q A^\dagger + L(x) A_\perp] \tag{A.7}$$

in which $L(x)$ *is an arbitrary matrix function with compatible size.*

PROOF. Noting that $\Lambda \frac{\partial S}{\partial x} A = -B^T P - K(x) C_Q$ and $\Lambda > 0$, (A.1) has a solution $\frac{\partial S}{\partial x}$ iff

$$\ker A \subset \ker(B^T P + K(x) C_Q). \tag{A.8}$$

Its equivalence to (A.6) is shown below. First, $\ker A \subset \ker C_Q$ is immediate since $Ax = 0$ implies $0 = x^T(A^T P + PA + C_Q^T C_Q)x = x^T C_Q^T C_Q x$. Now, suppose $x \in \ker A \subset \ker P$, then $C_Q x = 0$ also holds. So

$$(B^T P + K(x) C_Q)x = 0$$

is true, i.e. (A.6) implies (A.8). Conversely, When (A.8) is true, $Ax = 0$ leads to $C_Q x = 0$, $0 = (B^T P + K(x) C_Q)x = B^T P x$ and $0 = (A^T P + PA + C_Q^T C_Q)x = A^T P x$. That is, $(Px)^T [A \;\; B] = 0$. Then the controllability of (A, B) yields $Px = 0$, i.e. (A.8) implies (A.6). □

The following lemma provides a simple result on the partial derivative of a certain type of vector functions, which can be verified via straightforward computation. This lemma plays a key role in solving the PDME (A.7).

Lemma 3 *Let* $C \in \mathbb{R}^{q \times n}$ *be a constant matrix and* $S(x) \in \mathbb{R}^p$ *be a piecewise continuously differentiable vector function with* $S(0) = 0$*. Suppose that* $s_i(x)$*, the* i*th entry of* $S(x)$*, has the following form*

$$s_i(x) = \sum_{j=1}^{q} s_{ij}(c_j x), \quad i = 1, \ldots, p$$

in which c_j *denotes the* j*th row of the matrix* C*. Then*

$$\frac{\partial S}{\partial x} = \begin{bmatrix} s'_{11}(c_1 x) & \cdots & s'_{1q}(c_q x) \\ \vdots & & \vdots \\ s'_{p1}(c_1 x) & \cdots & s'_{pq}(c_q x) \end{bmatrix} C \tag{A.9}$$

holds where $s'_{ij}(c_j x) = ds_{ij}/d(c_j x)$*.*

Proof of Theorem 1. Owing to Lemma 2, (A.1) has been reduced to (A.7). First, let us find out the structures of matrix functions $K(x), L(x)$ in order for the matrix inequality constraint (A.2) to be satisfied. Substitution of (A.7) into (A.2) and completion of square with regard to $\tilde{K}(x) = K(x) - (C_Q A^\dagger B)^T$ yields

$$-L(x) A_\perp B - (A_\perp B)^T L^T(x) + \tilde{K}(x) \tilde{K}^T(x) \le 0 \tag{A.10}$$

owing to Lemma 1(5). Multiplication of matrix [2] $(A_\perp B)_\perp$ and its transpose to this inequality shows that $(A_\perp B)_\perp^T \tilde{K}(x) = 0$, which in turn yields $(A_\perp B)_\perp^T L(x) = 0$. That is. $L(x)$ and $\tilde{K}(x)$ have the following structures:

$$L(x) = (A_\perp B)^T G(x), \quad \tilde{K}(x) = (A_\perp B)^T F(x).$$

By (A.10), $G(x), F(x)$ are subject to the constraint:

$$-(A_\perp B)^T [G(x) + G^T(x) - F(x) F^T(x)] A_\perp B \le 0.$$

This is equivalent to $\Pi(x) \ge 0$ because $A_\perp B$ has full row rank. Meanwhile, the PDME (A.7) becomes

$$\begin{aligned} \frac{\partial S}{\partial x} = & -\Lambda^{-1}(A_\perp B)^T [F(x) C_Q A^\dagger + G(x) A_\perp] \\ & -\Lambda^{-1} B^T [P A^\dagger + (A^\dagger)^T Q A^\dagger]. \end{aligned} \tag{A.11}$$

According to Lemma 3, this equation is integrable when $G(x), F(x)$ are constrained to $G(x) = (g_{ij}(\alpha_j(x)))$ and constant F. Further, integration of (A.11) leads to the solution (23).

Finally, the first term in (A.11) disappears for nonsingular matrix A. Then the only solution becomes $S(x) = -\Lambda^{-1} B^T [P A^{-1} + A^{-T} Q A^{-1}] x = \Lambda^{-1} B^T A^{-T} P x$. □

B Proof of Theorem 2

The proof is based on Lyapunov stability theory. Lemma 1 guarantees the existence of $P \ge 0$ for Lyapunov equation $A^T P + PA + Q = 0$ under Assumption 2. So the continuously differentiable function

$$V(x) = x^T P x + 2 \sum_{i=1}^{m} \lambda_i \int_0^{s_i(x)} \phi_i(s) ds, \;\; \lambda_i > 0 \tag{B.1}$$

satisfies $V(x) \ge 0$. Further, according to Theorem 1 the PDMI (18) has a solution $S(x)$ when $\Pi(x) > 0$.

[2] Here $(A_\perp B)_\perp$ is defined as the matrix satisfying $(A_\perp B) \cdot (A_\perp B)_\perp = 0$.

Simple calculation shows that $\dot{V}(x)$ is equal to

$$\begin{bmatrix} x \\ \Phi \end{bmatrix}^T \begin{bmatrix} A^TP+PA & PB+A^T(\frac{\partial S}{\partial x})^T\Lambda \\ B^TP+\Lambda\frac{\partial S}{\partial x}A & \Lambda\frac{\partial S}{\partial x}B+B^T(\frac{\partial S}{\partial x})^T\Lambda \end{bmatrix} \begin{bmatrix} x \\ \Phi \end{bmatrix}. \tag{B.2}$$

So $\dot{V}(x) \le 0$ for all $x \in \mathbb{R}^n$ subject to the PDMI (18). Further, application of Proposition 1 shows that

$$\begin{aligned}\dot{V}(x) = & -(C_Qx+K^T(x)\Phi)^T(C_Qx+K^T(x)\Phi) \\ & +\Phi^T\big[\Lambda\frac{\partial S}{\partial x}B+B^T(\frac{\partial S}{\partial x})^T\Lambda+K(x)K^T(x)\big]\Phi\end{aligned}$$

So $\dot{V} \equiv 0$ iff $\big[\Lambda\frac{\partial S}{\partial x}B+B^T(\frac{\partial S}{\partial x})^T\Lambda+K(x)K^T(x)\big]\Phi \equiv 0$ and $C_Qx+K^T(x)\Phi \equiv 0$. This equivalent to $x \in \Omega$ because $K(x) = (C_QA^\dagger B)^T + (A_\perp B)^TF$, $-(A_\perp B)^T\Pi(x)A_\perp B = \Lambda\frac{\partial S}{\partial x}B+B^T(\frac{\partial S}{\partial x})^T\Lambda+K(x)K^T(x)$ and $\Pi(x) > 0$. According to LaSalle's invariance principle, x converges to the largest invariant set $\mathcal{V} \subset \Omega$.

$A_\perp B$ is nonsingular when $q = m$. Therefore, $\Phi(S(x)) \equiv 0$ is obtained in this case, i.e. $S(x) \equiv 0$. Also, $C_Qx \equiv 0$ holds true. So, the dynamics reduces to $\dot{x} = Ax$ and there holds $0 \equiv \dot{S}(x) = \frac{\partial S}{\partial x}\dot{x} = \frac{\partial S}{\partial x}Ax$. Applying (A.1) once again yields

$$(B^TP+K(x)C_Q)x = B^TPx \equiv 0.$$

Then, it follows from the Lyapunov equation $A^TP + PA + Q = 0$ and $Qx \equiv 0$ that $z(t) = Px(t)$ satisfies $\dot{z} = -A^Tz$ and $B^Tz \equiv 0$. Observability of $(B^T, -A^T)$ is a consequence of the controllability of (A, B), which leads to $z(t) = Px(t) \equiv 0$. From it, we obtain $x_\omega(t) \equiv 0, x_s(t) \equiv 0$ as well as $P_ox_o(t) \equiv 0$.

To show that $x_o(t) \equiv 0$ based on $S(x) \equiv 0$, we note that each input channel has exact one Jordan block with zero eigenvalue when $q = m$. To avoid clumsy notations, let us consider the following two input case. Other cases follow similarly.

$$A_o = \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right],\ B_o = \left[\begin{array}{cc} 1 & * \\ \hline 0 & 0 \\ 0 & 1 \end{array}\right] \ \Rightarrow\ A_{o\perp} = \left[\begin{array}{ccc} 1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right].$$

Decompose x_o as $[x_{o1}\ x_{o2}\ x_{o3}]^T$. $x_{o3}(t) \equiv 0$ is immediate due to $P_ox_o(t) \equiv 0$. Owing to the lower triangular structure of $G(x)$, we have

$$-\Lambda S(x) = \begin{bmatrix} 0 \\ r_2x_{o2} \end{bmatrix} + \begin{bmatrix} \int_0^{x_{o1}} g_{11}(w)dw \\ \int_0^{x_{o1}}[(*)g_{11}(w)+g_{21}(w)]dw \end{bmatrix}.$$

Then $x_{o1}(t) = 0$ follows from $g_{11}(w) > 0$, which in turn leads to $x_{o2}(t) = 0$.

Finally, we prove that $V(x)$ in (B.1) is radially unbounded if the algebraic multiplicity of every Jordan block with zero eigenvalue is no greater than two. To this end, we note that $V(x)$ can be expanded as

$$V(x) = x_o^TP_ox_o + x_\omega^TP_\omega x_\omega + x_s^TP_sx_s + \sum_{i=1}^m \int_0^{s_i}\phi_i(w)dw.$$

Due to $P_\omega > 0, P_s > 0$, we need only to show that x_o is contained in $V(x)$. Again, we look at the previous case. Then, $x_o^TP_ox_o = r_2x_{o3}^2$. Further, $s_1(x)$ contains x_{o1} and $s_2(x)$ contains x_{o2}. So $S(x)$ and hence $\int_0^{s_i}\phi_i(w)dw$ diverge to the infinity as $|x_o| \to \infty$. This completes the proof.