Chapter 7

Parametrization of Stabilizing Controllers

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Introduction

- In traditional control theories, no matter the classical control theory or the modern control theory, the central issue is to design a *single controller* capable of controlling the plant.
- Performance optimization problem basically boils down to shaping the closed-loop transfer matrix. However, in performance optimization it becomes an obstacle to ensure the stability of system.
- Question: is it possible to describe all controllers that stabilize the plant by a formula with a free parameter?
- Answer: YES!
- Parametrization of stabilizing controllers is a great progress in control theory.

A motivating example

Recall the 2-mass-spring system ($x = [\omega_M \ \phi \ \omega_L]^T$)

$$\dot{x} = \begin{bmatrix} -\frac{D_L}{J_L} & \frac{k}{J_L} & 0 \\ -1 & 0 & 1 \\ 0 & -\frac{k}{J_M} & -\frac{D_M}{J_M} \end{bmatrix} x + \begin{bmatrix} \frac{1}{J_L} \\ 0 \\ 0 \end{bmatrix} d + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{J_M} \end{bmatrix} u \qquad (1)$$

$$y_P = [0 \ \ 0 \ \ 1]x.$$

1. Performance spec: suppress the influence of load torque disturbance d, and ensure that ω_L tracks the reference input r.
2. Output to be controlled is the speed error $r - \omega_L$ of load, different from the measured signal ω_M
3. Torque disturbance d is different from the control input u in their properties and locations where they enter the system.
4. To optimize the disturbance (or reference tracking) response directly in control design, new input/output description is needed.

Generalized Feedback Control System

- Generalized plant $G(s)$: contains the plant, signals for performance optimization and weighting function.
- K: controller
- Input/output relationships

$$\begin{bmatrix} \hat{z}(s) \\ \hat{y}(s) \end{bmatrix} = G(s) \begin{bmatrix} \hat{w}(s) \\ \hat{u}(s) \end{bmatrix} \tag{2}$$

$$\hat{u}(s) = K(s)\hat{y}(s). \tag{3}$$

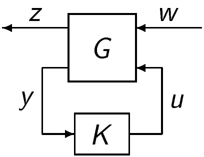
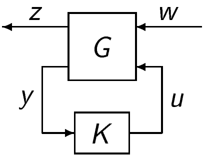


Figure: Generalized feedback system

Generalized plant

- Terms
  1. Performance output z: output vector used for specifying the control performance and model uncertainty
  2. Measured output y: input vector of the controller (for example, outputs of sensors, tracking errors, etc.)
  3. Disturbance w: external input vector used for specifying the control performance and model uncertainty
  4. Control input u: command vector of actuators
- Not only the design of feedback control systems, but also the design of feedforward systems like filters as well as the design of 2-DOF control systems can be handled in this framework.



Generalized plant

1. State equation

$$\begin{bmatrix} \dot{x} \\ z \\ y \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{bmatrix} \begin{bmatrix} x \\ w \\ u \end{bmatrix} \tag{4}$$

2. Partition in accordance with input $\begin{bmatrix} w \\ u \end{bmatrix}$ and output $\begin{bmatrix} z \\ y \end{bmatrix}$

$$G(s) = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right] \tag{5}$$

3. Closed-loop transfer matrix of $w \mapsto z$

$$H_{zw}(s) = G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}. \tag{6}$$

Example: 2-DOF control system

1. Plant output y_P and ref. input r used independently, instead of their difference $r - y_P$ as in 1-DOF control.
2. Capable of achieving the best tracking performance
3. $K(s) = [K_F \ K_B]$ contains two blocks $K_F(s)$ and $K_B(s)$, corresponding to the feedforward signal r and the feedback signal y_P resp.
4. Model of reference input: $W_R(s)$

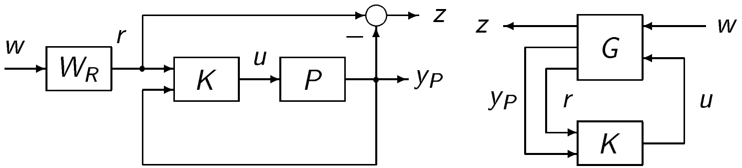


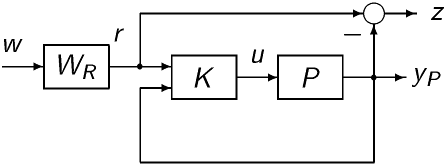
Figure: Reduction of 2-DOF system to generalized feedback system

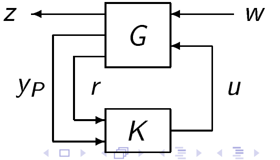
Example: 2-DOF control system

1. Performance output: tracking error $z = r - y_P$
2. Disturbance: impulse input w of $W_R(s)$
3. Measured output: $\begin{bmatrix} r \\ y_P \end{bmatrix}$

$$\begin{bmatrix} \hat{z} \\ \hline \hat{r} \\ \hat{y}_P \end{bmatrix} = G(s) \begin{bmatrix} \hat{w} \\ \hat{u} \end{bmatrix} = \left[\begin{array}{c|c} W_R & -P \\ \hline W_R & 0 \\ 0 & P \end{array}\right] \begin{bmatrix} \hat{w} \\ \hat{u} \end{bmatrix} \tag{7}$$

$$\hat{u} = K(s) \begin{bmatrix} \hat{r} \\ \hat{y}_P \end{bmatrix} = K_F \hat{r} + K_B \hat{y}_P \tag{8}$$





Example: 2-mass-spring system

$$\dot{x} = Ax + b_1 d + b_2 u \tag{9}$$

$$y_P = c_2 x.$$

1. Spec: load torque disturbance suppression
2. Performance output: tracking error of load speed

$$z = r - x_1 = [-1 \;\; 0 \;\; 0]x + r = c_1 x + r$$

3. Measured output (2-DOF): $[r \;\; y_P]^T$
4. Disturbance: $\overline{w} = [r \;\; d]^T$
5. Generalized plant: $[\overline{w}^T \;\; u]^T \mapsto [z \;\; y^T]^T$

$$P(s)\left\{\begin{bmatrix} \dot{x} \\ \hline z \\ \hline r \\ y_P \end{bmatrix} = \left[\begin{array}{c|cc|c} A & 0 & b_1 & b_2 \\ \hline c_1 & 1 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ c_2 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} x \\ \hline r \\ d \\ \hline u \end{bmatrix}\right. \tag{10}$$

2-mass-spring system: adding signal models

$$\begin{bmatrix} z \\ \hline r \\ y_P \end{bmatrix} = P(s) \begin{bmatrix} r \\ d \\ \hline u \end{bmatrix} \tag{11}$$

1. Models of ref input and disturbance: $W_R(s)$, $W_D(s)$
2. Generalized plant of $[w_1 \quad w_2 \quad u]^T \mapsto [z \quad y^T]^T$

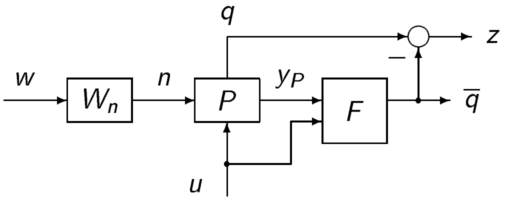
$$G(s) = P(s) \times \begin{bmatrix} W_R(s) & & \\ & W_D(s) & \\ & & 1 \end{bmatrix}. \tag{12}$$

Example: Filter design

1. Purpose: estimating a signal q from plant input and output
2. State equation of plant

$$\begin{aligned} \dot{x} &= Ax + B_1 n + B_2 u \\ y_P &= Cx + D_1 n + D_2 u \\ q &= Hx. \end{aligned}$$

3. Estimate $\overline{q}$: input/output signals (u, y_P) filtered by $F(s)$
4. Rule of filter design: minimizing the estimation error $z = q - \overline{q}$

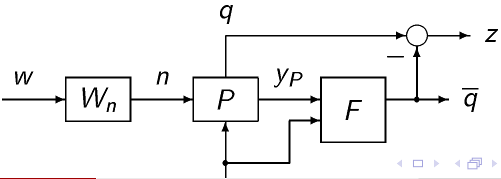


Example: Filter design

1. Disturbance: (n, u), Performance output: estimation error z, Measured output: (u, y_P), Control input: $\bar{q}$
2. Generalized plant

$$
\left[\begin{array}{c} \hat{z} \\ \hdashline \hat{y}_P \\ \hat{u} \end{array}\right] = \left[\begin{array}{c|cc:c} A & B_1 & B_2 & 0 \\ \hline H & 0 & 0 & -I \\ \hdashline C & D_1 & D_2 & 0 \\ 0 & 0 & I & 0 \end{array}\right] \left[\begin{array}{c} \hat{n} \\ \hat{u} \\ \hdashline \hat{\bar{q}} \end{array}\right] = P(s) \left[\begin{array}{c} \hat{n} \\ \hat{u} \\ \hdashline \hat{\bar{q}} \end{array}\right] \tag{13}
$$

$$
\hat{\bar{q}} = F(s) \begin{bmatrix} \hat{y}_P \\ \hat{u} \end{bmatrix}. \tag{14}
$$



Filter design: adding noise model

1. Colored noise n: $\hat{n}(s) = W_n(s)\hat{w}(s)$ and w is a white noise
2. Generalized plant with weighting function

$$\left[\begin{array}{c} \hat{z} \\ \hline \hat{y}_P \\ \hat{u} \end{array}\right] = G \left[\begin{array}{c} \hat{w} \\ \hat{u} \\ \hline \hat{q} \end{array}\right], \quad G = P \begin{bmatrix} W_n & & \\ & I & \\ & & I \end{bmatrix}. \tag{15}$$

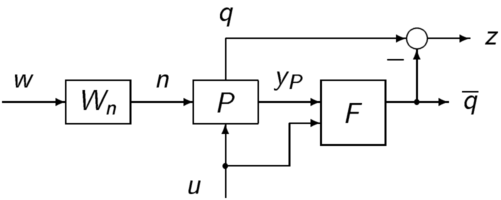


Figure: Filtering problem

Stable Plant Case

Theorem 1

Let $G(s)$ be stable. Then, all stabilizing controllers are parameterized by

$$K(s) = Q(I + G_{22}Q)^{-1}. \tag{16}$$

$Q(s)$: arbitrary stable matrix with compatible dimension.

(Proof) We need only prove that $K(s)$ stabilizes $G_{22}(s)$. That is,

$$(I - G_{22}K)^{-1},\ K(I - G_{22}K)^{-1},\ G_{22}K(I - G_{22}K)^{-1},\ (I - G_{22}K)^{-1}G_{22}$$

are all stable. These four transfer matrices are equal to

$$I + G_{22}Q,\ Q,\ G_{22}Q,\ (I + G_{22}Q)G_{22}$$

and are certainly stable.
Conversely, when $K(s)$ is a stabilizing controller, $K(I - G_{22}K)^{-1} := Q(s)$ must be stable. Solving for $K(s)$, we see that it is described by $K(s) = Q(I + G_{22}Q)^{-1}$. ●

Case of $G_{22}(s) = -P(s)$

Corollary 1

Assume that the plant $P(s)$ is stable. Then all controllers that stabilize the closed-loop system are parameterized by

$$K(s) = Q(I - PQ)^{-1}.$$

$Q(s)$: any stable matrix with appropriate dimension.

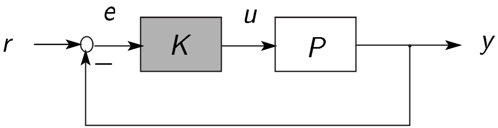


Figure: 1-DOF feedback system

Example 1

Consider the SISO feedback system where $P(s)$ is stable. Find all controllers that enable the asymptotic tracking of step ref input r.

(Solution) Laplace transform of tracking error

$$\hat{e}(s) = \hat{r}(s) - \hat{y}(s) = \frac{1}{1+PK}\hat{r}(s) = \frac{1}{1+PK}\frac{1}{s}.$$

Substitution of $K(s) = Q/(1-PQ)$ leads to $\hat{e}(s) = (1-PQ)\frac{1}{s}$.

$$e(\infty) = \lim_{s\to 0} s\hat{e}(s) = 1 - P(0)Q(0) = 0 \Rightarrow P(0) \neq 0, \quad Q(0) = \frac{1}{P(0)}.$$

Required controllers:

$$\left\{ K(s) = \frac{Q}{1-PQ} \middle| Q \text{ is stable and } Q(0) = \frac{1}{P(0)} \right\}.$$

$K(s)$ contains at least one integrator $1/s$ since

$$K(0) = \lim_{s\to 0} \frac{Q}{1-PQ} \to \infty.$$

For instance, for the plant

$$P(s) = \frac{1}{(s+1)(s+2)},$$

one of the controllers is obtained as

$$K(s) = \frac{2(s+1)(s+2)}{s(s+3)}$$

when the free parameter is selected as $Q = 1/P(0) = 2$. ∇

Example 2

Consider the SISO system. Assume that $P(s)$ is stable and $P(0) \neq 0$. Find all controllers that are capable of asymptotic rejection of step disturbance d. Further, for $P(s) = 1/(s+1)$, select the free parameter as $Q(s) = P^{-1}(s)\frac{k}{1+\epsilon s}$ $(\epsilon > 0)$ and design a controller satisfying $\|y\|_2 \leq 0.1$.

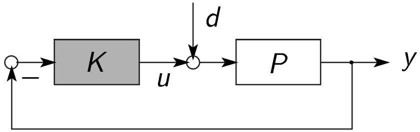


Figure: Disturbance control

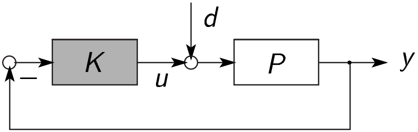


Figure: Disturbance control

(Solution) Disturbance response

$$\hat{y}(s) = \frac{P}{1+PK}\hat{d}(s) = \frac{P}{1+PK}\frac{1}{s}.$$

$K = Q/(1-PQ)$ yields

$$\hat{y}(s) = P(1-PQ)\frac{1}{s}. \tag{17}$$

All controllers guaranteeing zero steady-state output

$$\left\{ K = \frac{Q}{1-PQ} : Q(s) \text{ is stable and } Q(0) = \frac{1}{P(0)} \right\} \tag{18}$$

Each controller $K(s)$ contains at least one integrator $1/s$.

$\|y\|_2$ is bounded only if $y(\infty) = 0$. So, $k = Q(0) = 1/P(0) = 1$.

$$\hat{y}(s) = \frac{\epsilon}{(s+1)(\epsilon s+1)} = \frac{\epsilon}{1-\epsilon}\left(\frac{1}{s+1} - \frac{1}{s+1/\epsilon}\right)$$
$$\Rightarrow \; y(t) = \frac{\epsilon}{1-\epsilon}(e^{-t} - e^{-t/\epsilon}), \; t \geq 0$$

So,

$$\|y\|_2^2 = \int_0^\infty y^2(t)dt = \frac{\epsilon^2}{2(1+\epsilon)} \leq 0.1^2 \;\Rightarrow\; \epsilon^2 - 0.02\epsilon - 0.02 \leq 0.$$

Its solution is $-0.131 \leq \epsilon \leq 0.151$. Considering the stability condition $\epsilon > 0$, the final solution is $0 < \epsilon \leq 0.151$.
Obtained PI compensator:

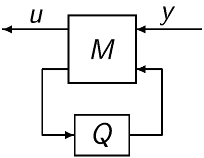
$$K(s) = \frac{s+1}{\epsilon s} = \frac{1}{\epsilon} + \frac{1}{\epsilon s}.$$

General Case

Theorem 2

Suppose that (A, B_2) is stabilizable and (C_2, A) is detectable. Let $A + B_2 F$ and $A + LC_2$ be stable. Then, all stabilizing controllers are given by the transfer matrix $\mathcal{F}_\ell(M, Q)$ from y to u, where $Q(s)$ is any stable matrix with an appropriate dimension.

$$M(s) = \left[\begin{array}{c|cc} A + B_2 F + LC_2 & -L & B_2 \\ \hline F & 0 & I \\ -C_2 & I & 0 \end{array}\right].$$



Outline of proof

Sufficiency: Set $Q(s) = (A_Q, B_Q, C_Q, D_Q)$.

$$K(s) = (A_K, B_K, C_K, D_K)$$

$$= \left[\begin{array}{cc|c} A + B_2F + LC_2 - B_2D_QC_2 & B_2C_Q & B_2D_QC_Q - L \\ -B_QC_2 & A_Q & B_Q \\ \hline F - D_QC_2 & C_Q & D_Q \end{array}\right] \tag{19}$$

A-matrix of closed-loop system H_{zw}

$$A_c = \begin{bmatrix} A + B_2D_QC_2 & B_2F - B_2D_QC_2 & B_2C_Q \\ B_2D_QC_2 - LC_2 & A + B_2F + LC_2 - B_2D_QC_2 & B_2C_Q \\ B_QC_2 & -B_QC_2 & A_Q \end{bmatrix} \tag{20}$$

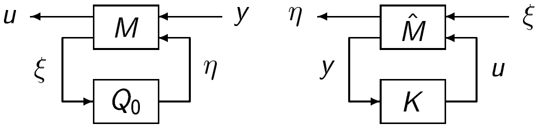
A_c is similar to the block triangular matrix:

$$\begin{bmatrix} A + B_2F & B_2C_Q & B_2F - B_2D_QC_2 \\ 0 & A_Q & -B_QC_2 \\ 0 & 0 & A + LC_2 \end{bmatrix} \tag{21}$$

This matrix obviously is stable.

Outline of proof

Necessity: we need just prove that any stabilizing controller $K(s)$ can be described as $K(s) = \mathcal{F}_\ell(M, Q_0)$ with a stable $Q_0(s)$.



Input/output relation

$$\begin{bmatrix} \hat{u} \\ \hat{\xi} \end{bmatrix} = M(s) \begin{bmatrix} \hat{y} \\ \hat{\eta} \end{bmatrix}, \ \hat{u} = K(s)\hat{y}; \quad \begin{bmatrix} \hat{\eta} \\ \hat{y} \end{bmatrix} = \hat{M}(s) \begin{bmatrix} \hat{\xi} \\ \hat{u} \end{bmatrix}, \ \hat{\eta} = Q_0(s)\hat{\xi}.$$

Relationship between $\hat{M}$ and M

$$\hat{M}(s) = \begin{bmatrix} & I \\ I & \end{bmatrix} M^{-1} \begin{bmatrix} & I \\ I & \end{bmatrix}.$$

Outline of proof

State realization of $\hat{M}(s)$:

$$\hat{M}(s) = \left[\begin{array}{c|cc} A & -L & B_2 \\ \hline -F & 0 & I \\ C_2 & I & 0 \end{array}\right].$$

$\hat{M}(s)$ and $G(s)$ share the same (2, 2) block, namely $\hat{M}_{22}(s) = G_{22}(s) = C_2(sI - A)^{-1}B_2$. So they both are stabilized by $K(s)$. Therefore, $Q_0(s) := \mathcal{F}_\ell(\hat{M}, K)$ is stable.

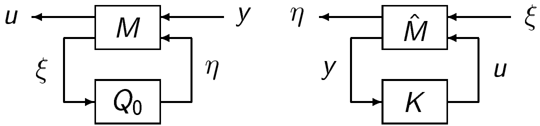


Figure: Input/output relations of $K = \mathcal{F}_\ell(M, Q_0)$ and $Q_0 = \mathcal{F}_\ell(\hat{M}, K)$

Stabilization of integrator $P(s) = 1/s := G_{22}$

A state realizations is $(0, 1, 1, 0)$. When $F = L = -1$ are chosen, $A + B_2F = A + LC_2 = -1$ are stable. From the coefficient matrix

$$M(s) = \left[\begin{array}{c|cc} -2 & 1 & 1 \\ \hline -1 & 0 & 1 \\ -1 & 1 & 0 \end{array}\right] = \frac{1}{s+2}\left[\begin{array}{cc} -1 & s+1 \\ s+1 & -1 \end{array}\right],$$

we get

$$K(s) = -\frac{1}{s+2} + \left(\frac{s+1}{s+2}\right)^2 Q(s)\left(1 + \frac{1}{s+2}Q(s)\right)^{-1}.$$

When $Q(s) = 0$, the controller is $K(s) = -1/(s+2)$.
Characteristic polynomial of CLS is equal to $s(s+2) + 1 = (s+1)^2$, so CLS is stable.

Youla Parametrization

$$\left[\begin{array}{c|cc} A+B_2F & B_2 & -L \\ \hline F & I & 0 \\ C_2 & 0 & I \end{array}\right] := \begin{bmatrix} D(s) & -Y(s) \\ N(s) & -X(s) \end{bmatrix} \tag{22}$$

$$\left[\begin{array}{c|cc} A+LC_2 & -B_2 & L \\ \hline F & I & 0 \\ C_2 & 0 & I \end{array}\right] := \begin{bmatrix} -\tilde{X}(s) & \tilde{Y}(s) \\ -\tilde{N}(s) & \tilde{D}(s) \end{bmatrix} \tag{23}$$

Theorem 3

Suppose that (A, B_2) is stabilizable and (C_2, A) is detectable, $A + B_2F$ and $A + LC_2$ are stable. Then,

(1) $G_{22}(s) = N(s)D^{-1}(s) = \tilde{D}^{-1}(s)\tilde{N}(s)$;

(2) *All controllers are parameterized by*

$$K(s) = (\tilde{X} - Q\tilde{N})^{-1}(\tilde{Y} - Q\tilde{D}) = (Y - DQ)(X - NQ)^{-1} \tag{24}$$

Affine Structure in Controller Parameter

Controller $K(s)$

$$\begin{bmatrix} \dot{x}_K \\ u \end{bmatrix} = \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix} \begin{bmatrix} x_K \\ y \end{bmatrix}. \tag{25}$$

Closed-loop system

$$\begin{bmatrix} \dot{x} \\ \dot{x}_K \\ \hline z \end{bmatrix} = \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} \begin{bmatrix} x \\ x_K \\ \hline w \end{bmatrix} \tag{26}$$

$$\begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} = \left[\begin{array}{cc|c} A + B_2 D_K C_2 & B_2 C_K & B_1 + B_2 D_K D_{21} \\ B_K C_2 & A_K & B_K D_{21} \\ \hline C_1 + D_{12} D_K C_2 & D_{12} C_K & D_{11} + D_{12} D_K D_{21} \end{array}\right]. \tag{27}$$

Affine Structure in Controller Parameter

Relationship between the coefficient matrices of closed-loop system and controller

$$\begin{aligned} A_c &= \begin{bmatrix} A + B_2 D_K C_2 & B_2 C_K \\ B_K C_2 & A_K \end{bmatrix} \\ &= \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} B_2 D_K C_2 & B_2 C_K \\ B_K C_2 & A_K \end{bmatrix} \\ &= \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} B_2 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} D_K & C_K \\ B_K & A_K \end{bmatrix} \begin{bmatrix} C_2 & 0 \\ 0 & I \end{bmatrix}. \end{aligned}$$

A_c is an affine function of the coefficient matrix of controller:

$$\mathcal{K} = \begin{bmatrix} D_K & C_K \\ B_K & A_K \end{bmatrix}. \tag{28}$$

Affine Structure in Controller Parameter

$$\left[\begin{array}{cc} A_c & B_c \\ C_c & D_c \end{array}\right] = \left[\begin{array}{cc} \overline{A} & \overline{B}_1 \\ \overline{C}_1 & \overline{D}_{11} \end{array}\right] + \left[\begin{array}{c} \overline{B}_2 \\ \overline{D}_{12} \end{array}\right] \mathcal{K}[\overline{C}_2, \ \overline{D}_{21}] \tag{29}$$

$$\left[\begin{array}{ccc} \overline{A} & \overline{B}_1 & \overline{B}_2 \\ \overline{C}_1 & \overline{D}_{11} & \overline{D}_{12} \\ \overline{C}_2 & \overline{D}_{21} & \end{array}\right] = \left[\begin{array}{cc|c|cc} A & 0 & B_1 & B_2 & 0 \\ 0 & 0 & 0 & 0 & I \\ \hline C_1 & 0 & D_{11} & D_{12} & 0 \\ \hline C_2 & 0 & D_{21} & & \\ 0 & I & 0 & & \end{array}\right]. \tag{30}$$

1. Closed-loop transfer matrix is a nonlinear function of the controller. Meanwhile, in state space their coefficient matrices have an affine relation which is much simpler.
2. It is because of this affine feature that the state space method is effective in various kinds of optimal control designs.
3. In the $\mathcal{H}_\infty$ control and multiple-objective control, this affine relationship plays a fundamental role in deriving the LMI solutions.

Affine Structure in Free Parameter

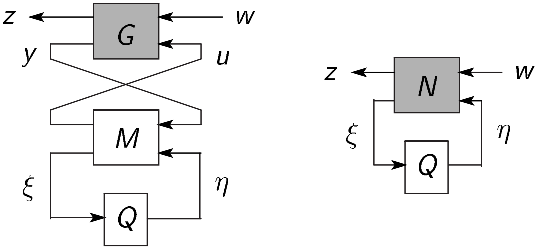


Figure: Closed-loop system

Some notations:

$$\begin{aligned} A_F &:= A + B_2 F, \quad C_F := C_1 + D_{12} F \\ A_L &:= A + L C_2, \quad B_L := B_1 + L D_{21} \\ \hat{A} &:= A + B_2 F + L C_2. \end{aligned} \tag{31}$$

Affine Structure in Free Parameter

Closed-loop transfer matrix $w \mapsto z$:

$$H_{zw}(s) = \mathcal{F}_\ell(G, K) = \mathcal{F}_\ell(G, \mathcal{F}_\ell(M, Q)) = \mathcal{F}_\ell(N, Q)$$

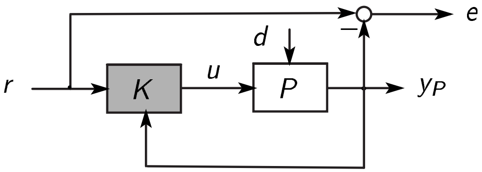
$$N(s) = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} = \left[\begin{array}{cc|cc} A_F & -B_2F & B_1 & B_2 \\ 0 & A_L & B_L & 0 \\ \hline C_F & -D_{12}F & D_{11} & D_{12} \\ 0 & C_2 & D_{21} & 0 \end{array}\right], \quad N_{22}(s) = 0. \tag{32}$$

Eventually, the closed-loop transfer matrix becomes

$$H_{zw}(s) = N_{11}(s) + N_{12}(s)Q(s)N_{21}(s). \tag{33}$$

Namely, $H_{zw}(s)$ is an affine function of $Q(s)$. This affine structure will be used in solving the $\mathcal{H}_2$ optimal control problem.

Structure of 2-Degree-of-Freedom Systems



- Plant dynamics

$$\dot{x} = Ax + Hd + Bu \tag{34}$$

$$y_P = Cx \tag{35}$$

- Performance output

$$e(t) = r(t) - y_P(t) \tag{36}$$

- Disturbance d may enter the closed-loop system at a location of different from control input u (for instance, 2-mass-spring system), so their coefficient matrices are set differently.

Structure of 2-Degree-of-Freedom Systems

- Transfer matrices $P_u(s): \ u \mapsto y_P$, $P_d(s): \ d \mapsto y_P$

$$P_u(s) = C(sI - A)^{-1}B, \quad P_d(s) = C(sI - A)^{-1}H \tag{37}$$

- Partition of free parameter $Q(s)$

$$Q(s) = [Q_F(s) \ \ Q_B(s)] \tag{38}$$

- Ref tracking T_{er}, disturbance suppression T_{ed}

$$T_{er}(s) = I + N_{12}(s)Q_F(s) \tag{39}$$

$$\begin{aligned} T_{ed}(s) =& N_{12}(s)Q_B(s)C(sI - A_L)^{-1}H \\ & - N_{12}(s)F(sI - A_L)^{-1}H - C(sI - A_F)^{-1}H. \end{aligned} \tag{40}$$

- $T_{er}(s) \propto Q_F(s)$, $T_{ed}(s) \propto Q_B(s)$
  $T_{er}(s)$ and $T_{ed}(s)$ can be designed independently.
- Stable plant case

$$T_{er}(s) = I - P_u(s)Q_F(s), \quad T_{ed}(s) = -P_u(s)Q_B(s)P_d(s) - P_d(s) \tag{41}$$

Design example

- 1st-order system

$$\dot{x} = -2x + u + d, \quad y_P = 2x$$

- Ref input r and the disturbance d are unit step signal $1(t)$.
- Control spec: reduce the reference tracking error $e(t)$
- Plant is stable and

$$P_u(s) = P_d(s) = \frac{2}{s+2}.$$

- Free parameters chosen as

$$Q_F(s) = P_u^{-1}(s)\frac{1}{\epsilon s + 1}, \; Q_B(s) = -P_u^{-1}(s)\frac{1}{\tau s + 1}, \; \epsilon, \tau > 0$$

$$\Rightarrow T_{er}(s) = 1 - P_u Q_F = \frac{s}{s + 1/\epsilon}$$

$$T_{ed}(s) = -(P_u Q_B + 1)P_d = -\frac{2s}{(s+2)(s+1/\tau)}.$$

Design example

- Tracking error

$$\hat{e}(s) = T_{er}\hat{r} + T_{ed}\hat{d} = \frac{1}{s+1/\epsilon} - \frac{2}{(s+2)(s+1/\tau)}$$
$$\Rightarrow e(t) = e^{-t/\epsilon} - \frac{2\tau}{1-2\tau}\left(e^{-2t} - e^{-t/\tau}\right). \quad (42)$$

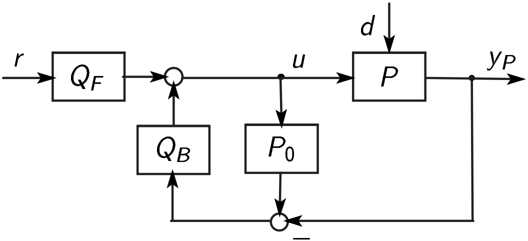
- Tracking error can be reduced by lowering ϵ, τ
- Controller

$$K(s) = \frac{Q}{1+QG_{22}} = \frac{[Q_F \quad Q_B]}{1+Q_B P_u} = \frac{\tau s+1}{\tau s}\left[\frac{s+2}{2(\epsilon s+1)} \quad -\frac{s+2}{2(\tau s+1)}\right].$$

- Low frequency gain of $K(s)$ increases when τ is reduced, while ϵ does not affect the low frequency gain of $K(s)$.
- To realize signal tracking using an input as small as possible, we should better mainly use feedforward control (that is, lowering ϵ only).
- Feedback should be strengthened only when the disturbance is strong (lowering both ϵ and τ).

Implementation 1

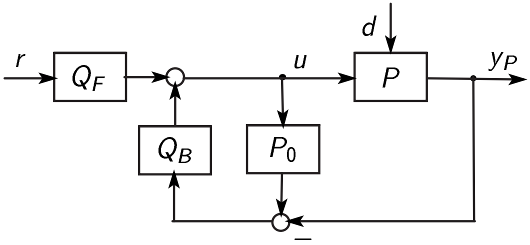
$$K := [K_F \quad -K_B] = \frac{[Q_F \quad -Q_B]}{1+[Q_F \quad -Q_B]G_{22}} = \frac{[Q_F \quad -Q_B]}{1-PQ_B}. \tag{43}$$



- Feature: input of $Q_B(s)$ becomes zero when $P = P_0$ and $d(t) = 0$. So feedback controller $K_B(s)$ is not activated.
- Transfer function $r \mapsto y_P$

$$H_{y_P r}(s) = P(s)Q_F(s)$$

Implementation 1



- Model-matching: let closed-loop transfer function match or close to a reference model $M(s)$ with good performance
- Feedforward compensator Q_F

$$PQ_F = M \Rightarrow Q_F(s) = \frac{M(s)}{P(s)} \tag{44}$$

- Q_F must be stable. So when the plant $P(s)$ have unstable zeros, the model $M(s)$ must also contain the same zeros. That is, for a non-minimum phase plant, the output response cannot be improved

Implementation 2

- Physical implication: feedback controller is activated when output of actual plant is different from and that of ref model $M(s)$; when they are the same, the feedback controller stops working.

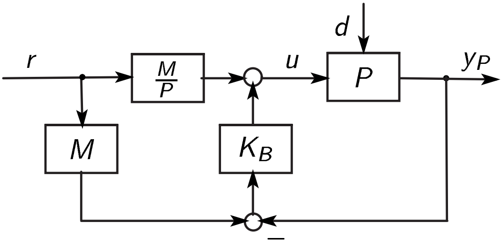


Figure: Another Form of 2-DOF Systems

Example: plant with low damping

$$P(s) = \frac{4}{s^2 + s + 4} \quad (\zeta = \frac{1}{4},\ \omega_n = 2).$$

1. Ref model with a strengthened damping:

$$M(s) = \frac{4}{s^2 + 3s + 4} \quad (\zeta^* = 0.75,\ \omega_n^* = 2).$$

2. Feedforward compensator

$$Q_F(s) = \frac{M}{P} = \frac{s^2 + s + 4}{s^2 + 3s + 4}.$$

3. Feedback controller

$$Q_B(s) = P^{-1}\frac{1}{(\epsilon s + 1)^2} \Rightarrow K_B(s) = \frac{s^2 + s + 4}{2\epsilon^2 s(s + 2/\epsilon)}.$$

4. Sensitivity function

$$S(s) = \frac{1}{1 + PK_B} = 1 - PQ_B = 1 - \frac{1}{(\epsilon s + 1)^2} = \frac{\epsilon s(\epsilon s + 2)}{(\epsilon s + 1)^2}$$