Chapter 19

Regional Pole Placement

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Motivation

1. In the modern control theory, the so-called pole placement is to place the poles to fixed points in the complex plane.
2. However, it is impossible to fix the closed-loop poles to specific points when the system has uncertainty.
3. Nevertheless, it is still possible to place the closed-loop poles in a region.
4. From the viewpoint of robust performance, the response quality of the CLS is guaranteed if the closed-loop poles can be locked in a prescribed region.

Performance and Pole Location

1. Prototype 2nd-order system

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \quad 0 < \zeta < 1 \tag{1}$$

2. Poles of $G(s)$: $p = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$
3. Rise time is proportional to $1/\omega_n$, so ω_n must be greater than a certain number $r > 0$.
4. $R > |p|$ necessary to avoid large input.
5. Therefore,

$$r \leq |p| = \omega_n \leq R.$$

6. Let the damping ratio corresponding to the greatest allowable overshoot be ζ_p, the damping ratio must satisfy

$$\zeta \geq \zeta_p.$$

Performance and Pole Location

1. angle between the poles and the real axis must satisfy

$$\tan\theta = \frac{\sqrt{1-\zeta^2}}{\zeta} = \sqrt{\frac{1}{\zeta^2} - 1} \leq \sqrt{\frac{1}{\zeta_p^2} - 1}$$

$$\Rightarrow \theta \leq \theta_p := \arctan\sqrt{\frac{1}{\zeta_p^2} - 1}.$$

2. To shorten the settling time, we need, w.r.t. the required convergence rate σ

$$\Re(p) = -\zeta\omega_n \leq -\sigma.$$

Basically this can be ensured by adjusting r.

Performance and Pole Location

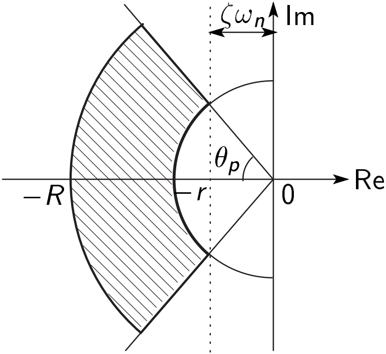
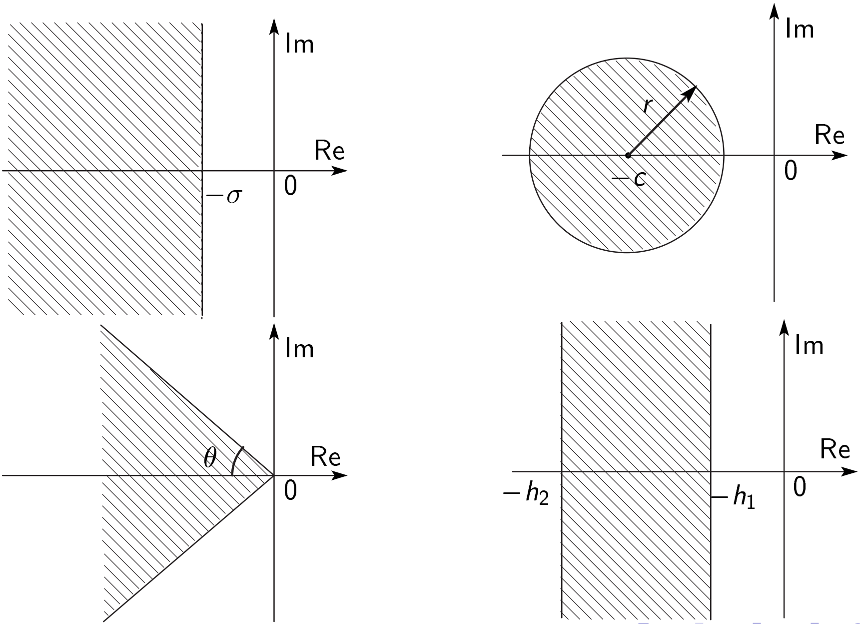


Figure: Desirable pole region for 2nd-order systems


Im
Re
$-\sigma$
0
Im
Re
r
$-c$
0
Im
Re
θ
0
Im
Re
$-h_2$
$-h_1$
0

Characterization of LMI Region

- Half-plane: $\mathbb{R}(s) < -\sigma$

$$x < -\sigma \Leftrightarrow z + \overline{z} < -2\sigma. \tag{2}$$

- Disk: centered at $(-c,\ 0)$ and with a radius r

$$\begin{aligned}\overline{(z+c)}(z+c) < r^2 &\Leftrightarrow -r - (\overline{z}+c)\cdot\frac{1}{-r}\cdot(z+c) < 0\\ &\Leftrightarrow \begin{bmatrix} -r & z+c \\ \overline{z}+c & -r \end{bmatrix} < 0\\ &\Leftrightarrow \begin{bmatrix} -r & c \\ c & -r \end{bmatrix} + z\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \overline{z}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} < 0. \end{aligned} \tag{3}$$

Characterization of LMI Region

- Sector: $\left|\arg z - \pi\right| < \theta$

$$\frac{|y|}{-x} < \tan\theta \Leftrightarrow x\sin\theta < -|y|\cos\theta < 0$$
$$\Leftrightarrow (x\sin\theta)^2 > (y\cos\theta)^2, \quad x\sin\theta < 0$$
$$\Leftrightarrow \begin{bmatrix} x\sin\theta & jy\cos\theta \\ -jy\cos\theta & x\sin\theta \end{bmatrix} < 0.$$

Substitution of $x = (z+\bar{z})/2$, $jy = (z-\bar{z})/2$ leads to

$$\begin{bmatrix} (z+\bar{z})\sin\theta & (z-\bar{z})\cos\theta \\ -(z-\bar{z})\cos\theta & (z+\bar{z})\sin\theta \end{bmatrix} < 0 \Leftrightarrow$$
$$z\begin{bmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{bmatrix} + \bar{z}\begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix} < 0. \tag{4}$$

Definition of LMI Region

1. LMI region D: the set of complex number z characterized by

$$D = \{z \in \mathbb{C} | \, f_D(z) < 0\} \tag{5}$$
$$f_D(z) = L + zM + \bar{z}M^T. \tag{6}$$

2. Matrix $f_D(z)$: the *characteristic function*
3. L and M are both square matrices.

Condition for Regional Pole Placement

1. System
$$\dot{x} = Ax. \tag{7}$$
2. Characteristic function
$$f_D(z) = L + zM + \bar{z}M^T \tag{8}$$
3. Characteristic matrix
$$M_D(A, X) = L \otimes X + M \otimes (AX) + M^T \otimes (AX)^T. \tag{9}$$
4. Replacement relation between $f_D(z)$ and $M_D(A, X)$
$$(1,\ z,\ \bar{z}) \Leftrightarrow (X,\ AX,\ (AX)^T). \tag{10}$$

Theorem 1

All eigenvalues of matrix A are located in an LMI region D iff $\exists X$ such that
$$M_D(A, X) < 0. \tag{11}$$

Example 1

1. *Consider a disk centered at $(-c, 0)$ and with radius r.*
2. *Characteristic function*

$$f_D(z) = \begin{bmatrix} -r & c \\ c & -r \end{bmatrix} + z \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \bar{z} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

$$\Rightarrow L = \begin{bmatrix} -r & c \\ c & -r \end{bmatrix}, \quad M = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

3. *Condition*

$$\begin{aligned} M_D(A, X) &= L \otimes X + M \otimes (AX) + M^T \otimes (AX)^T \\ &= \begin{bmatrix} -rX & cX \\ cX & -rX \end{bmatrix} + \begin{bmatrix} 0 & AX \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ (AX)^T & 0 \end{bmatrix} \\ &= \begin{bmatrix} -rX & cX + AX \\ cX + (AX)^T & -rX \end{bmatrix} < 0. \end{aligned} \tag{12}$$

Relation between $f_D(z)$ and $M_D(A, X)$

- Replacement relation between $f_D(z)$ and $M_D(A, X)$:

$$(1,\ z,\ \bar{z}) \Leftrightarrow (X,\ AX,\ (AX)^T). \tag{13}$$

Example 2

1. *Sector in the left half plane and with an angle θ:*

$$f_D(z) = z \begin{bmatrix} \sin\theta & \cos\theta \\ -\cos\theta & x\sin\theta \end{bmatrix} + \bar{z} \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}.$$

2. *Condition for pole location of $\dot{x} = Ax$:*

$$M_D(A, X) = \begin{bmatrix} (AX + XA^T)\sin\theta & (AX - XA^T)\cos\theta \\ -(AX - XA^T)\cos\theta & (AX + XA^T)\sin\theta \end{bmatrix} < 0. \tag{14}$$

Example 3

$$A = \begin{bmatrix} 0 & 1 \\ -10 & -6 \end{bmatrix}, \quad \lambda(A) = -3 \pm j.$$

- *Eigenvalues of matrix A are contained in 3 regions:*

  (1) disk: $c = 0,\ r = 5$; (2) half plane: $\sigma = 2$; (3) sector: $\theta = \pi/4$
- *Solutions to the corresponding LMIs*

$$X_1 = \begin{bmatrix} 0.1643 & -0.2756 \\ -0.2756 & 0.8687 \end{bmatrix} > 0, \ X_2 = \begin{bmatrix} 0.1567 & -0.3532 \\ -0.3532 & 1.0106 \end{bmatrix} > 0$$

$$X_3 = \begin{bmatrix} 0.1335 & -0.2187 \\ -0.2187 & 0.4799 \end{bmatrix} > 0.$$

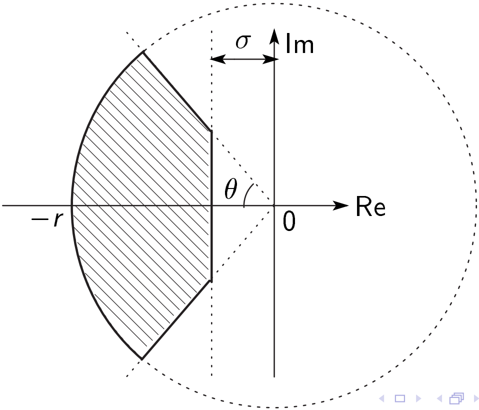
  The same conclusion is obtained.

- *But when $\sigma = 4$, system poles are not in the half plane. (11) has no positive definite solution.*

Composite LMI Region

Corollary 1

All eigenvalues of matrix A are in a composite region $D_1 \cap D_2$ iff $\exists X > 0$ satisfying $M_{D_1}(A, X) < 0$ and $M_{D_2}(A, X) < 0$.



Example 4

Eigenvalues of system matrix

$$A = \begin{bmatrix} 0 & 1 \\ -10 & -6 \end{bmatrix}$$

are located in the following 3 regions respectively:

$$\textit{(1) disk: } c = 0,\ r = 5; \textit{ (2) half plane: } \sigma = 2; \textit{ (3) sector: } \theta = \pi/4$$

Intersection of these 3 regions is the shaded part of the figure on the preceding slide. Solving for the common solution of those three LMIs, we obtain

$$X = \begin{bmatrix} 18.357 & -38.4586 \\ -38.4586 & 120.2923 \end{bmatrix} > 0.$$

Hence the same conclusion is obtained.

Feedback Controller Design

1. Plant

$$\dot{x} = Ax + Bu, \quad y = Cx \tag{15}$$

2. Controller

$$\dot{x}_K = A_K x_K + B_K y, \quad u = C_K x_K + D_K y \tag{16}$$

3. Closed-loop system

$$\begin{bmatrix} \dot{x} \\ \dot{x}_K \\ z \end{bmatrix} = \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} \begin{bmatrix} x \\ x_K \\ w \end{bmatrix} \tag{17}$$

in which

$$A_c = \begin{bmatrix} A + B_2 D_K C_2 & B_2 C_K \\ B_K C_2 & A_K \end{bmatrix}.$$

Feedback Controller Design

Design condition

$$M_D(A_c, P) = L \otimes P + M \otimes (A_c P) + M^T \otimes (A_c P)^T < 0 \qquad (18)$$

As $P = \Pi_2 \Pi_1^{-1}$, multiplication of Π_1^T, Π_1 from both sides yields

$$L \otimes (\Pi_1^T P \Pi_1) + M \otimes (\Pi_1^T A_c P \Pi_1) + M^T \otimes (\Pi_1^T A_c P \Pi_1)^T < 0. \qquad (19)$$

Variable change

$$\mathbb{A} = N A_K M^T + N B_K C X + Y B C_K M^T + Y(A + B D_K C) X$$
$$\mathbb{B} = N B_K + Y B D_K,\ \mathbb{C} = C_K M^T + D_K C X,\ \mathbb{D} = D_K \qquad (20)$$
$$\Pi_1^T P \Pi_1 = \Pi_1^T \Pi_2 = \begin{bmatrix} X & I \\ I & Y \end{bmatrix},\ \Pi_1^T A_c P \Pi_1 = \begin{bmatrix} AX + B\mathbb{C} & A + B\mathbb{D}C \\ \mathbb{A} & YA + \mathbb{B}C \end{bmatrix} \qquad (21)$$

Feedback Controller Design

Final design condition:

$$L \otimes \begin{bmatrix} X & I \\ I & Y \end{bmatrix} + M \otimes \begin{bmatrix} AX + B\mathbb{C} & A + B\mathbb{D}C \\ \mathbb{A} & YA + \mathbb{B}C \end{bmatrix} + M^T \otimes \begin{bmatrix} AX + B\mathbb{C} & A + B\mathbb{D}C \\ \mathbb{A} & YA + \mathbb{B}C \end{bmatrix}^T < 0. \tag{22}$$

In addition, $P > 0$ is equivalent to

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0. \tag{23}$$

Design Example: Mass-Spring System

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -100 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u.$$

Composite region:

(1) disk: $c = 0, \;\; r = 5$; (2) half plane: $\sigma = 2$; (3) sector: $\theta = \pi/4$

Solution to Eqs. (22) and (23), all positive definite

$$X = \begin{bmatrix} 85.6357 & -189.7711 \\ -189.7711 & 629.9055 \end{bmatrix}, \; Y = \begin{bmatrix} 629.9055 & -189.7711 \\ -189.7711 & 85.6357 \end{bmatrix}$$

Controller

$$K(s) = \frac{90s^2 + 1087s + 13023}{s^2 + 12.2859s - 128.9044}.$$

Poles of the closed-loop system

$$-2.8568 \pm j1.1716, \quad -3.2861 \pm j1.7570$$

All are located in the three specified LMI regions.

Polytpoic System

Mass-spring-damper system ($u = 0$)

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} x.$$

$p_1 = k/m, \; p_2 = b/m$. Then

$$\dot{x} = \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + p_1 \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} + p_2 \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \right\} x.$$

A is an affine function of parameter vector $p = [p_1 \;\; p_2]^T$.

General form

$$\dot{x} = A(p)x, \quad x(0) \neq 0. \tag{24}$$

Robust Regional Pole Placement

Corollary 2

All eigenvalues of matrix $A(p)$ are located in LMI region D iff $\exists X(p) > 0$ satisfying

$$M_D(A(p), X(p)) < 0. \quad (25)$$

1. $X(p)$ depends on parameter vector p. However, nobody knows the relationship between $X(p)$ and p so far.
2. We follow the philosophy of quadratic stability and fix $X(p)$ as a constant matrix.
3. Sufficient condition:

$$M_D(A(p), X) < 0 \quad (26)$$

has a real positive definite solution.

Simplification

Polytopic system

$$A(p) = \sum_{i=1}^{N} p_i A_i, \quad \sum_{i=1}^{N} p_i = 1, \; p_i \geq 0. \tag{27}$$

Corollary 3

For a matrix polytope $A(p)$, if $\exists X > 0$ satisfying

$$M_D(A_i, X) < 0 \quad \forall i = 1, \ldots, N, \tag{28}$$

then the eigenvalues of $A(p)$ are all located in the LMI region D.

Example: mass-spring-damper system

- $m = 1$, $k = 3$, friction b

$$A = \begin{bmatrix} 0 & 1 \\ -3 & -b \end{bmatrix}, \; b > 0.$$

- LMI region: disk centered at $(-c, 0) = (-2.5, 0)$ and with a radius $r = 2$.
- Eigenvalues of matrix A are $\frac{-b \pm \sqrt{b^2 - 12}}{2}$.
  1. Complex roots ($b < \sqrt{12}$), the condition is

$$r^2 > \left(c - \frac{b}{2}\right)^2 + \left(\frac{\sqrt{12 - b^2}}{2}\right)^2 \;\Rightarrow\; b > \frac{c^2 - r^2 + 3}{c} = 2.1.$$

  2. Real roots, the condition becomes

$$c + r > \frac{d + \sqrt{b^2 - 12}}{2} \;\Rightarrow\; b < c + r + \frac{3}{c + r} = 5.167.$$

- Corollary 3 succeeds only when $b \in [2.3, \; 4]$, conservative.

Norm-Bounded Parametric System

$$\dot{x} = A_\Delta x = (A + B\Delta(I - D\Delta)^{-1}C)x, \quad \|\Delta(t)\|_2 \leq 1. \tag{29}$$

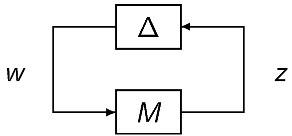
Equivalent to the CLS made up by a nominal system $M(s)$

$$M \begin{cases} \dot{x} = Ax + Bw \\ z = Cx + Dw \end{cases} \tag{30}$$

and a norm-bounded parameter uncertainty $\Delta(t)$

$$w = \Delta z, \quad \|\Delta(t)\|_2 \leq 1.$$

In addition, as $\Delta(t)$ varies arbitrarily in the range of $\|\Delta(t)\|_2 \leq 1$, the matrix $I - D\Delta$ is invertible iff $\|D\|_2 < 1$.



A Sufficient Condition

Theorem 2

If there exist matrix $P > 0, \; Q > 0$ *satisfying LMI*

$$\begin{bmatrix} N_D(A,P) & M_1 \otimes (PB) & (M_2 Q) \otimes C^T \\ M_1^T \otimes (B^T P) & -Q \otimes I & Q \otimes D^T \\ (QM_2^T) \otimes C & Q \otimes D & -Q \otimes I \end{bmatrix} < 0 \tag{31}$$

$$N_D(A,P) = L \otimes P + M \otimes (PA) + M^T \otimes (PA)^T, \tag{32}$$

then poles of the uncertain system (29) are all located inside the region D.

An Example

1. LMI region: disk with characteristic function

$$f_D(z) = \begin{bmatrix} -r & c \\ c & -r \end{bmatrix} + z \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \bar{z} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

2. Condition

$$\begin{bmatrix} -rP & cP + PA & PB & 0 \\ cP + A^T P & -rP & 0 & C^T \\ B^T P & 0 & -I & D^T \\ 0 & C & D & -I \end{bmatrix} < 0. \tag{33}$$

On Polytopic Systems

1. System with only one uncertain parameter

$$\dot{x} = (\theta_1 A_1 + \theta_2 A_2)x + Bu, \ y = Cx; \quad \theta_1, \theta_2 \geq 0, \ \theta_1 + \theta_2 = 1. \quad (34)$$

2. Controller

$$\dot{x}_K = A_K x_K + B_K y, \quad u = C_K x_K + D_K y, \quad (35)$$

3. Condition for pole placement in an LMI region D:

$$L \otimes \begin{bmatrix} X & I \\ I & Y \end{bmatrix} + \mathrm{He}\left\{ M \otimes \begin{bmatrix} AX + B\mathbb{C} & A + B\mathbb{D}C \\ \mathbb{A} & YA + \mathbb{B}C \end{bmatrix} \right\} < 0$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0$$

have a solution.

On Polytopic Systems

1. Unknown variables

$$\mathbb{A} = NA_K M^T + NB_K CX + YBC_K M^T + Y(\theta_1 A_1 + \theta_2 A_2 + BD_K C)X$$
$$\mathbb{B} = NB_K + YBD_K,\ \mathbb{C} = C_K M^T + D_K CX,\ \mathbb{D} = D_K.$$

2. Although $\mathbb{A} = \theta_1 \mathbb{A}_1 + \theta_2 \mathbb{A}_2$ in which

$$\mathbb{A}_1 = NA_K M^T + NB_K CX + YBC_K M^T + Y(A_1 + BD_K C)X$$
$$\mathbb{A}_2 = NA_K M^T + NB_K CX + YBC_K M^T + Y(A_2 + BD_K C)X,$$

it is not guaranteed that they have a common solution when calculating A_K from $(\mathbb{A}_1, \mathbb{A}_2)$.

On Polytopic Systems

1. However, when θ_1, θ_2 are known, we may use a controller with
$$A_K = \theta_1 A_{K1} + \theta_2 A_{K2}.$$
2. In this case,
$$\begin{aligned}\mathbb{A}_1 &= NA_{K1}M^T + NB_K CX + YBC_K M^T + Y(A_1 + BD_K C)X \\ \mathbb{A}_2 &= NA_{K2}M^T + NB_K CX + YBC_K M^T + Y(A_2 + BD_K C)X\end{aligned}$$
and $(A_{K1},\ A_{K2})$ can be inversely calculated from $(\mathbb{A}_1,\ \mathbb{A}_2)$.
3. This is the so-called gain-scheduling method to be introduced in the next chapter.

Design for Norm-Bounded Parametric System

1. Norm-bounded parametric system

$$G\begin{cases} \dot{x} = Ax + B_1 w + B_2 u \\ z = C_1 x + D_{11} w + D_{12} u \\ y = C_2 x + D_{21} w \end{cases} \tag{36}$$

$$w = \Delta z, \quad \|\Delta\|_2 \leq 1. \tag{37}$$

2. Controller K:

$$\begin{aligned} \dot{x}_K &= A_K x_K + B_K y \\ u &= C_K x_K + D_K y \end{aligned} \tag{38}$$

3. Task: robustly place the closed-loop poles inside an LMI region D

Sufficient Condition

$$\begin{bmatrix} L \otimes (\Pi_1^T \Pi_2) + \mathrm{He}\{M \otimes (\Pi_2^T A_c \Pi_1)\} & M_1 \otimes (\Pi_2^T B_c) & M_2 \otimes (\Pi_1^T C_c^T) \\ M_1^T \otimes (B_c^T \Pi_2) & -I & I \otimes D_c^T \\ M_2^T \otimes (C_c \Pi_1) & I \otimes D_c & -I \end{bmatrix} < 0 \tag{39}$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0. \tag{40}$$

have a solution, in which

$$\Pi_1^T \Pi_2 = \begin{bmatrix} X & I \\ I & Y \end{bmatrix}, \quad \Pi_2^T A_c \Pi_1 = \begin{bmatrix} AX + B_2 \mathbb{C} & A + B_2 \mathbb{D} C_2 \\ \mathbb{A} & YA + \mathbb{B} C_2 \end{bmatrix}$$

$$\Pi_2^T B_c = \begin{bmatrix} B_1 + B_2 \mathbb{D} D_{21} \\ YB_1 + \mathbb{B} D_{21} \end{bmatrix}, \quad C_c \Pi_1 = [C_1 X + D_{12} \mathbb{C} \quad C_1 + D_{12} \mathbb{D} C_2]$$

$$\mathbb{A} = NA_K M^T + NB_K C_2 X + YB_2 C_K M^T + Y(A + B_2 D_K C_2)X$$

$$\mathbb{B} = NB_K + YB_2 D_K, \ \mathbb{C} = C_K M^T + D_K C_2 X, \ \mathbb{D} = D_K$$

Robust Design Example: Mass-Spring-Damper System

- $m = 1$ kg, $b \in [0,\ 2]$ Ns/m, $k \in [80,\ 120]$ N/m

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -k & -b \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad y = [1 \ \ 0]x.$$

- Uncertain parameters

$$k = k_0(1 + w_1\delta_1),\ b = b_0(1 + w_2\delta_2), \quad |\delta_i| \le 1$$

$$k_0 = 100,\ b_0 = 1,\ w_1 = \frac{k_{\max}}{k_0} - 1 = 0.2,\ w_2 = \frac{b_{\max}}{b_0} - 1 = 1.$$

- Normalizing the uncertain matrix $\Delta = [\delta_1 \ \ \delta_2]$, we get

$$A = \begin{bmatrix} 0 & 1 \\ -k_0 & -b_0 \end{bmatrix},\ B_1 = B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ C_1 = -\sqrt{2}\begin{bmatrix} k_0 w_1 & 0 \\ 0 & b_0 w_2 \end{bmatrix}$$

$$C_2 = [1 \ \ 0],\ D_{11} = D_{12} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ D_{21} = 0$$

1. Design spec: place the poles of CLS inside the intersection of a disk ($c = 0$, $r = 15$) and a half plane ($\sigma = 1$)
2. Controller
$$K(s) = \frac{-66s^2 - 405s + 13315}{s^2 + 30.415s + 277.62}.$$
3. CLS poles (all located in the given region)
   - Nominal
$$(-6.0189 \pm j8.717, -9.6886 \pm j5.9056)$$
   - Four vertices
$$(-9.3103 \pm j7.3826, -5.8972 \pm j5.3125)$$
$$(-9.2397 \pm j9.562, -6.9678 \pm j1.3258)$$
$$(-3.9167 \pm j10.3592, -11.2458 \pm j6.0098)$$
$$(-5.8157 \pm j10.5058, -10.3918 \pm j5.5414)$$
4. $K(s)$ is very sensitive to noise because $K(\infty) = -66$
5. Noise reduction should be supplemented, a multiobjective design.