Chapter 11

Major Uncertainty Models

Table of contents

Cruise control

1. Simplified speed control model for a car

$$P(s) = \frac{1}{Ms + \mu} \tag{1}$$

2. M: mass of the car, μ: coefficient of road friction.
3. The mass changes with load and the friction coefficient changes with the road condition. In system design, we only know the ranges of these parameters:

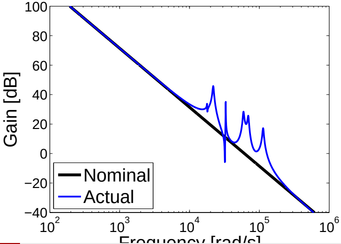
$$M_1 \leq M \leq M_2, \quad \mu_1 \leq \mu \leq \mu_2.$$

IEEJ HDD benchmark

1. Physical model of HDD
$$\tilde{P}(s) = \frac{K_p}{s^2} + \frac{A_1}{s^2 + 2\zeta_1\omega_1 s + \omega_1^2} + \frac{A_1}{s^2 + 2\zeta_1\omega_2 s + \omega_2^2} + \cdots \qquad (2)$$
2. Obtained via finite element method and modal analysis
3. High order resonant modes vary with manufacturing error, hundreds of thousands of HDDs controlled by the same controller.
4. Control design carried out based on rigid body model $P(s) = K_p/s^2$

IEEJ HDD benchmark

i	f_i (Hz)	ζ_i	A_i
1	4100 ($\pm$15%)	0.02	-1.0
2	8200 ($\pm$15%)	0.02	1.0
3	12300 ($\pm$10%)	0.02	-1.0
4	16400 ($\pm$10%)	0.02	1.0
5	3000 ($\pm$5%)	0.005	0.01 ($-200\% \sim 0\%$)
6	5000 ($\pm$5%)	0.001	0.03 ($-200\% \sim 0\%$)
K_p		3.744×10^9	



Philosophy of Robust Control

1. Since a real physical system cannot be modeled accurately, it is impossible to describe a real system using a single transfer function!
2. Instead, we can determine a model called the *nominal plant*, then evaluate the difference between the real system and the model, i.e., the uncertainty range.
3. In this way, we can obtain a set of systems that includes the real system.
4. If the stability and control performance are guaranteed for the plant set, these properties carry over to the real system.
5. In other words, the achievable performance will be significantly limited by the worst-case uncertainty because we need to maintain the same level of performance for all plants in the set.
6. Important are the description method of plant set and the modeling of uncertainty bounds.

Category of Model Uncertainty

1. Parameter Uncertainty
2. Dynamic Uncertainty
   - Unmodeled high freq resonant modes, as shown in the figure.
   - Dynamics ignored deliberately for the simplification of system analysis and design, particularly the high freq dynamics.

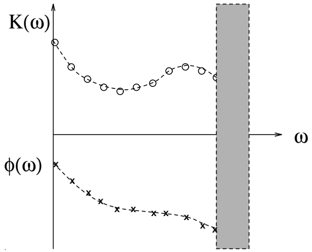


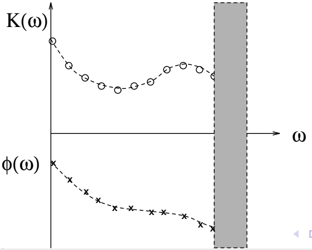
Figure: Identified freq response with uncertainty in high freq

System identification and unmodeled uncertainty

1. Typical identification method: apply sinusoidal input to the system, then measure the steady-state response of the output.
2. For input $\sin(\omega t)$, the steady-state response is still a sinusoid with amplitude $K(\omega)$ and phase angle $\phi(\omega)$:

$$K(\omega) = |G(j\omega)|, \quad \phi(\omega) = \arg G(j\omega). \tag{3}$$

3. Changing the input freq, the freq response at a different freq can be measured. Repeating this process, a set of gain-phase data is obtained.



System identification and unmodeled uncertainty

1. A transfer function is identified by finding a rational function whose freq response matches the measured data.
2. However, in reality vibration causes mechanical attrition, so the input freq cannot be too high. It is impossible to get the freq response in the high freq band.
3. Model uncertainty in the high freq band is inevitable.

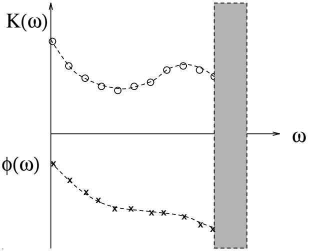


Figure: Identified freq response with uncertainty in high freq

Example: $\Delta(s) = P(s) - P_0(s)$

1. Identification experiments under different conditions yields a number of models $P_i(j\omega)$ $(i = 1, 2, \cdots)$.
2. By calculating the gains of all $P_i(j\omega) - P_0(j\omega)$, we can find a bounding function $W(s)$ satisfying

$$|\Delta_i(j\omega)| = |P_i(j\omega) - P_0(j\omega)| \leq |W(j\omega)| \quad \forall\omega,\ i. \tag{4}$$

3. This function is called the *weighting function*.
4. $W(s)$ is usually chosen as a low order stable rational function.
5. Assuming that *the set with bound $W(s)$ is filled with uncertainty Δ*, the uncertainty can be expressed as ($\delta(s)$: normalized uncertainty)

$$\Delta(s) = W(s)\delta(s),\ |\delta(j\omega)| \leq 1\ \forall\omega. \tag{5}$$

6. As such, we have obtained a plant set:

$$\{P(s) \mid P = P_0 + W\delta,\ \|\delta\|_\infty \leq 1\}. \tag{6}$$

7. It is natural to assume that the real plant is contained in this set.

Plant Set with Additive Uncertainty

$$P(s) = P_0 + \Delta W, \quad \|\Delta\|_\infty \leq 1. \tag{7}$$

1. $P_0(s)$: nominal plant
2. $\Delta(s)$: normalized uncertainty
3. $W(s)$: weighting function that bounds the uncertainty set

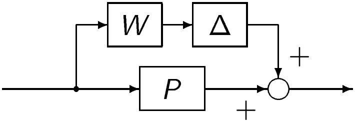


Figure: Plant set with additive uncertainty

Plant Set with Multiplicative Uncertainty

$$P(s) = (1 + \Delta W)P_0, \quad \|\Delta\|_\infty \leq 1 \tag{8}$$

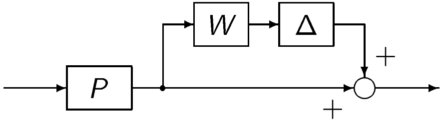


Figure: Plant set with multiplicative uncertainty

Plant Set with Feedback Uncertainty

Type I

$$P(s) = \frac{P_0}{1 + \Delta W P_0}, \quad \|\Delta\|_\infty \leq 1, \tag{9}$$

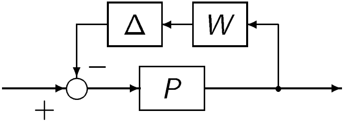


Figure: Plant set with Type I feedback uncertainty

Plant Set with Feedback Uncertainty

Type II

$$P(s) = \frac{P_0}{1 + \Delta W}, \quad \|\Delta\|_\infty \leq 1. \tag{10}$$

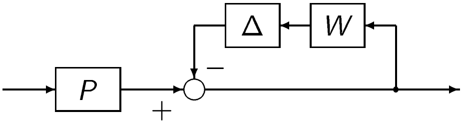
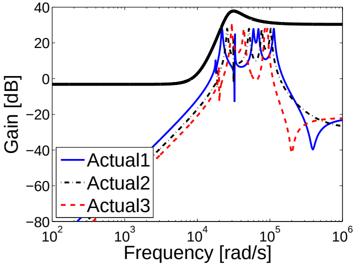


Figure: Plant set with Type II feedback uncertainty

Example: HDD

1. Model the high freq resonant modes of HDD as a multiplicative uncertainty.
2. Draw the relative error $\left|1 - \frac{P(j\omega)}{P_0(j\omega)}\right|$ in a Bode plot.
3. Determine a minimum phase weighting function s.t. the gain of its freq response covers the relative errors. An example is the high-pass transfer function shown by the solid line.

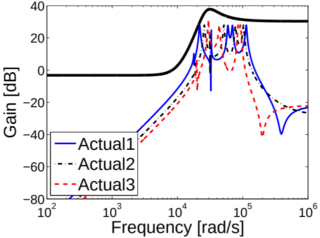


Example: HDD

1. Real plant $P(s)$ is contained in the plant set:

$$P(s) = P_0(1 + \Delta W), \quad P_0(s) = \frac{k}{s^2}, \quad \|\Delta\|_\infty \le 1.$$

2. Region below $W(s)$ (solid line) is treated as the uncertainty region, but the real uncertainty comprises only a small part of it. This inevitably enlarges the plant set and brings conservatism into robustness condition. The payoff is that the description of uncertainty is quite simple and suitable for analysis and design.



System Identification Case

1. Calculate the difference between the freq responses of true plant $P(j\omega)$ and the nominal model $P_0(j\omega)$ on Bode plot (solid line)
2. Find a weighting function $W(s)$ (dashed line) which covers $|P(j\omega) - P_0(j\omega)|$

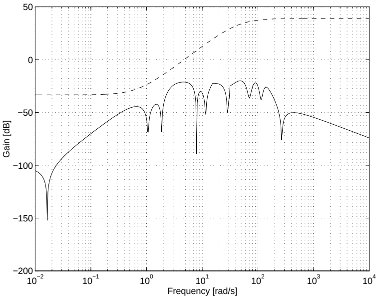


Figure: Weighting function of uncertainty

Model Reduction Case

1. This is the case where a high order plant $P(s)$ (including time-delay) is approximated by a low order model $P_0(s)$.
2. A weighting function $W(s)$ is determined on Bode plot that satisfies

$$|P(j\omega) - P_0(j\omega)| < |W(j\omega)| \quad \text{(additive uncertainty)}$$

or

$$\left|1 - \frac{P(j\omega)}{P_0(j\omega)}\right| < |W(j\omega)| \quad \text{(multiplicative uncertainty).}$$

3. Concretely speaking, we draw the curve of the left side of the inequality, then find a rational transfer function $W(s)$ that covers this curve.

Parametric system: mechanical example

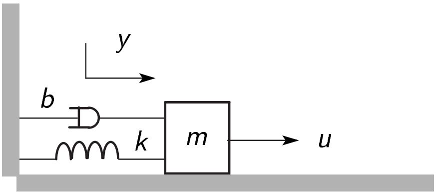
1. Mass-damper-spring system: displacement $y(t)$, external force $u(t)$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u, \quad x = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}. \tag{11}$$

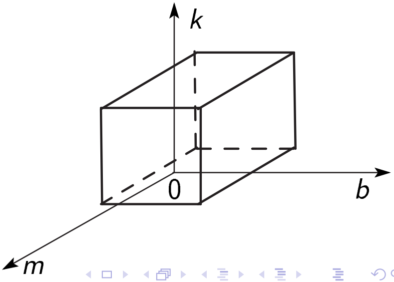
2. Uncertain parameters (m, b, k) take values in:

$$m_1 \leq m \leq m_2, \quad b_1 \leq b \leq b_2, \quad k_1 \leq k \leq k_2. \tag{12}$$

3. Vector $[m \ \ b \ \ k]$ forms a hexahedron with 8 vertices in 3D space.



(a) System configuration

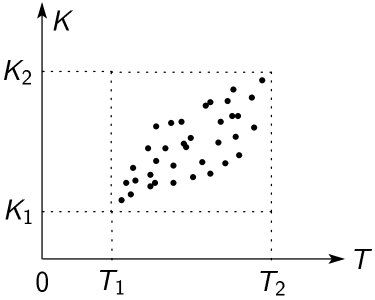


Parametric system: electrical example

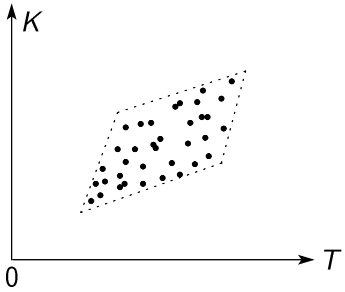
1. Linear motor: use Lorentz force to offer a straight drive
2. Operating principle: place many pairs of magnets with the same polarity in the order of S, N respectively on two parallel rails, then change the polarity of electromagnetic windings in an order of $S \to N \to S \to N$ periodically so as to generate a Lorentz force to drive the stage straight forward.
3. Model of linear motor
$$P(s) = \frac{K}{s(Ts+1)} = \frac{K/T}{s(s+1/T)}. \tag{13}$$
4. State equation (displacement and speed of the stage as the states)
$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{1}{T} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{K}{T} \end{bmatrix} u. \tag{14}$$
5. Linear motor has nonlinearities such as magnetic flux leakage and nonlinear friction with rails. Cannot be described accurately by a linear model.

Parametric system: electrical example

- By doing numerous experiments w.r.t. different speed commands we can obtain many pairs of (T, K).
- Relation between T and K is not clear. A smart way is to enclose the experimental data using a minimum rectangle (Figure (a)).
- Measured data may be enclosed tightly with a polygon (Figure (b)).



(a) Rectangle approximation



(b) Tighter polytopic approximation

Polytopic Set of Parameter Vectors

How should we describe the uncertain parameter vector $[m \quad b \quad k]$?

1. One parameter case: mass m takes values in an interval $[m_1, m_2]$
2. m_1 and m_2, the two ends of the interval, are known. The question is how to use these vertices to express an arbitrary point in the interval.
3. Simplest way: take a vertex as a starting point, then add the variation relative to this starting point. This variation is expressed as product of $m_2 - m_1$ and a factor $\lambda \in [0, 1]$ which represents the variation rate.
4. $m \in [m_1, m_2]$ can be written as

$$m = m_2 - \lambda(m_2 - m_1) = \lambda m_1 + (1 - \lambda)m_2$$

5. More compact form: setting $\alpha_1 = \lambda, \alpha_2 = 1 - \lambda$

$$m = \alpha_1 m_1 + \alpha_2 m_2, \ \alpha_1 + \alpha_2 = 1, \ \alpha_i \geq 0. \tag{15}$$

6. This equation expresses a *convex combination* of the vertices m_1, m_2.

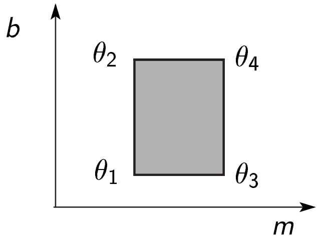
Polytopic Set of Parameter Vectors

1. Two-parameter case: $m \in [m_1, m_2]$ and $b \in [b_1, b_2]$

$$m = \alpha_1 m_1 + \alpha_2 m_2, \ \alpha_1 + \alpha_2 = 1, \ \alpha_i \geq 0$$
$$b = \beta_1 b_1 + \beta_2 b_2, \ \beta_1 + \beta_2 = 1, \ \beta_i \geq 0$$

2. Vector $[m \quad b]^T$ forms a rectangle with 4 vertices (Figure (a)):

$$\theta_1 = \begin{bmatrix} m_1 \\ b_1 \end{bmatrix}, \ \theta_2 = \begin{bmatrix} m_1 \\ b_2 \end{bmatrix}, \ \theta_3 = \begin{bmatrix} m_2 \\ b_1 \end{bmatrix}, \ \theta_4 = \begin{bmatrix} m_2 \\ b_2 \end{bmatrix}. \quad (16)$$



(a) Two uncertain parameters

Polytopic Set of Parameter Vectors

1. Parameter vector can be written as

$$\theta = \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} (\beta_1 + \beta_2)(\alpha_1 m_1 + \alpha_2 m_2) \\ (\alpha_1 + \alpha_2)(\beta_1 b_1 + \beta_2 b_2) \end{bmatrix}$$
$$= \alpha_1\beta_1 \begin{bmatrix} m_1 \\ b_1 \end{bmatrix} + \alpha_1\beta_2 \begin{bmatrix} m_1 \\ b_2 \end{bmatrix} + \alpha_2\beta_1 \begin{bmatrix} m_2 \\ b_1 \end{bmatrix} + \alpha_2\beta_2 \begin{bmatrix} m_2 \\ b_2 \end{bmatrix}.$$

2. Renaming the coefficient of each vertex

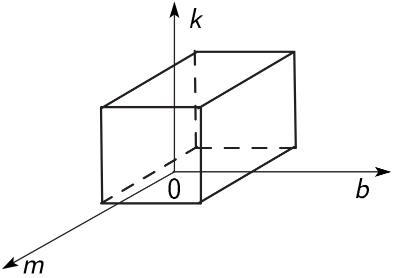
$$\lambda_1 = \alpha_1\beta_1,\ \lambda_2 = \alpha_1\beta_2,\ \lambda_3 = \alpha_2\beta_1,\ \lambda_4 = \alpha_2\beta_2 \ \Rightarrow\ \lambda_i \geq 0$$
$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = \alpha_1(\beta_1 + \beta_2) + \alpha_2(\beta_1 + \beta_2) = \alpha_1 + \alpha_2 = 1.$$

3. That is, a vector θ in a rectangle can always be described as a convex combination of the four vertices:

$$\theta = \lambda_1\theta_1 + \lambda_2\theta_2 + \lambda_3\theta_3 + \lambda_4\theta_4. \tag{17}$$

Polytopic Set of Parameter Vectors

1. Three-parameter case: parameter vector set forms a hexahedron with 8 vertices



(b) Three uncertain parameters

Polytopic Set of Parameter Vectors

1. Convex combination of all vertices:

$$\begin{bmatrix} m \\ b \\ k \end{bmatrix} = \lambda_1 \begin{bmatrix} m_1 \\ b_1 \\ k_1 \end{bmatrix} + \lambda_2 \begin{bmatrix} m_1 \\ b_1 \\ k_2 \end{bmatrix} + \lambda_3 \begin{bmatrix} m_1 \\ b_2 \\ k_1 \end{bmatrix} + \lambda_4 \begin{bmatrix} m_1 \\ b_2 \\ k_2 \end{bmatrix} + \lambda_5 \begin{bmatrix} m_2 \\ b_1 \\ k_1 \end{bmatrix} + \lambda_6 \begin{bmatrix} m_2 \\ b_1 \\ k_2 \end{bmatrix} + \lambda_7 \begin{bmatrix} m_2 \\ b_2 \\ k_1 \end{bmatrix} + \lambda_8 \begin{bmatrix} m_2 \\ b_2 \\ k_2 \end{bmatrix} \tag{18}$$

   $\lambda_i \geq 0$ for all i and $\sum_{i=1}^{8} \lambda_i = 1$.

2. A point in a polytope can always be expressed as a convex combination of all vertices. This conclusion applies to parameter vectors with any dimension.

Matrix Polytope and Polytopic System

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u, \tag{19}$$

1. Parameters appear in forms of product, ratio and reciprocal.
2. Question 1: can the coefficient matrix be expressed as a convex combination of the matrices obtained at the vertices?
3. Mass m appears only as a reciprocal $1/m$ and can be expressed as
$$\frac{1}{m} = \alpha_1 \frac{1}{m_1} + \alpha_2 \frac{1}{m_2}, \ \alpha_1 + \alpha_2 = 1, \ \alpha_i \geq 0.$$
4. Substitution of this equation yields
$$\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} = \alpha_1 \begin{bmatrix} 0 & 1 \\ -\frac{k}{m_1} & -\frac{b}{m_1} \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & 1 \\ -\frac{k}{m_2} & -\frac{b}{m_2} \end{bmatrix}$$
$$\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} = \alpha_1 \begin{bmatrix} 0 \\ \frac{1}{m_1} \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ \frac{1}{m_2} \end{bmatrix}$$

Matrix Polytope and Polytopic System

1. Question 2: can the product $b \times \frac{1}{m}$ still be expressed as a convex combination of vertices?
2. YES!

$$\begin{aligned}\frac{b}{m} &= (\alpha_1 \frac{1}{m_1} + \alpha_2 \frac{1}{m_2})(\beta_1 b_1 + \beta_2 b_2) \\ &= \alpha_1 \beta_1 \frac{b_1}{m_1} + \alpha_1 \beta_2 \frac{b_2}{m_1} + \alpha_2 \beta_1 \frac{b_1}{m_2} + \alpha_2 \beta_2 \frac{b_2}{m_2} \\ &= \lambda_1 \frac{b_1}{m_1} + \lambda_2 \frac{b_2}{m_1} + \lambda_3 \frac{b_1}{m_2} + \lambda_4 \frac{b_2}{m_2}\end{aligned}$$

   $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1, \lambda_i \geq 0.$
3. So, the matrices can be described as matrix polytopes:

$$A(m, b) = \lambda_1 A(b_1, m_1) + \lambda_2 A(b_2, m_1) + \lambda_3 A(b_1, m_2) + \lambda_4 A(b_2, m_2)$$
$$B(m) = \lambda_1 B(m_1) + \lambda_2 B(m_1) + \lambda_3 B(m_2) + \lambda_4 B(m_2).$$

4. The same is true for the parameter product $k \times \frac{1}{m}$.

Power issue

1. More question: can the square of a parameter still be expressed as a convex combination of its vertices?
2. NO!

$$m^2 = (\alpha_1 m_1 + \alpha_2 m_2)^2 = \alpha_1^2 m_1^2 + 2\alpha_1 \alpha_2 m_1 m_2 + \alpha_2^2 m_2^2.$$

   Cross product $m_1 m_2$ appears and m^2 cannot be described only by m_1^2 and m_2^2.
3. In robust control design, this power problem will be a bottleneck.

Conclusion

1. In summary, if θ belongs to a polytope and its power higher than 2 does not exist, then a system with uncertain parameter vector θ can be described by

$$\dot{x} = A(\theta)x + B(\theta)u \tag{20}$$

$$y = C(\theta)x \tag{21}$$

2. Each coefficient matrix is a matrix polytope:

$$A(\theta) = \sum_{i=1}^{N} \lambda_i A(\theta_i), \; B(\theta) = \sum_{i=1}^{N} \lambda_i B(\theta_i), \quad C(\theta) = \sum_{i=1}^{N} \lambda_i C(\theta_i)$$

$$\lambda_i \geq 0, \; \sum_{i=1}^{N} \lambda_i = 1. \tag{22}$$

   θ_i is a known vertex of parameter vector.

3. This sort of uncertain systems are called *polytopic systems*.

Norm-bounded Parametric System

Mass-spring-damper system:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u.$$

1. Set the nominal parameters as (m_0, k_0, b_0) and

$$\frac{k}{m} = \frac{k_0}{m_0}(1+w_1\delta_1), \ \frac{b}{m} = \frac{b_0}{m_0}(1+w_2\delta_2), \ \frac{1}{m} = \frac{1}{m_0}(1+w_3\delta_3), \ |\delta_i| \le 1.$$

2. Then

$$\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k_0}{m_0} & -\frac{b_0}{m_0} \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} [\delta_1 \quad \delta_2 \quad \delta_3] \begin{bmatrix} \frac{k_0}{m_0} w_1 & 0 \\ 0 & \frac{b_0}{m_0} w_2 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{m_0} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [\delta_1 \quad \delta_2 \quad \delta_3] \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m_0} w_3 \end{bmatrix},$$

Norm-bounded Parametric System

1. State equation can be rewritten as

$$\dot{x} = (A + B_1 \Delta C_1)x + (B_2 + B_1 \Delta D_{12})u$$

2. $\Delta = [\delta_1 \quad \delta_2 \quad \delta_3]$ and

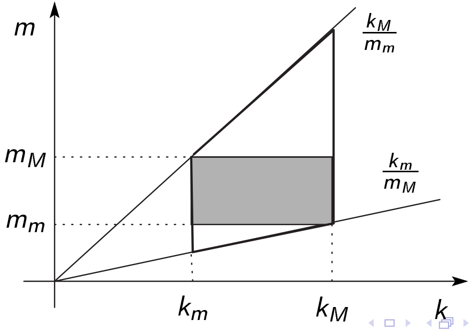
$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k_0}{m_0} & -\frac{b_0}{m_0} \end{bmatrix}, \; B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0 \\ \frac{1}{m_0} \end{bmatrix}$$

$$C_1 = -\begin{bmatrix} \frac{k_0}{m_0} w_1 & 0 \\ 0 & \frac{b_0}{m_0} w_2 \\ 0 & 0 \end{bmatrix}, \; D_{12} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m_0} w_3 \end{bmatrix}.$$

3. Size of uncertainty vector Δ is measured by norm.
4. This kind system is called a *norm-bounded parametric system*.

Norm-bounded Parametric System: a better model

1. Defect of the present model: uncertainty range is enlarged
2. Look at $\frac{k}{m}$ and $\frac{1}{m}$. Suppose $k_{\mathrm{m}} \le k \le k_{\mathrm{M}}$, $m_{\mathrm{m}} \le m \le m_{\mathrm{M}}$
3. Parameter range is the shaded square.
4. But $\alpha = \frac{k}{m}, \frac{1}{m}$ is treated as uncertain parameters.
5. Range of α becomes $\frac{k_{\mathrm{m}}}{m_{\mathrm{M}}} \le \alpha \le \frac{k_{\mathrm{M}}}{m_{\mathrm{m}}}$ and the parameter region is enlarged to that enclosed by solid lines.



Descriptor form

$$\begin{bmatrix} 1 & 0 \\ 0 & m \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & 1 \\ -k & -b \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$m = m_0(1 + w_1\delta_1), \; k = k_0(1 + w_2\delta_2), \; b = b_0(1 + w_3\delta_3), \; |\delta_i| \leq 1.$$

1. Dividing the second row of state equation by m_0, we get

$$(I + B_1\Delta C_1)\dot{x} = (A + B_1\Delta C_2)x + B_2 u$$
$$\Leftrightarrow \dot{x} = (I + B_1\Delta C_1)^{-1}(A + B_1\Delta C_2)x + (I + B_1\Delta C_1)^{-1}B_2 u$$

2. $\Delta = [\delta_1 \quad \delta_2 \quad \delta_3]$ and

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k_0}{m_0} & -\frac{b_0}{m_0} \end{bmatrix}, \; B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0 \\ \frac{1}{m_0} \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 0 & w_1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \; C_2 = -\begin{bmatrix} 0 & 0 \\ \frac{k_0}{m_0}w_2 & 0 \\ 0 & \frac{b_0}{m_0}w_3 \end{bmatrix}.$$

LFT description

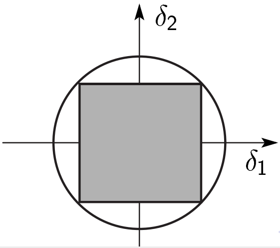
1. Set $v = C_1\dot{x}$, $z = -v + C_2x$ and $w = \Delta z$.
2. Then,

$$\dot{x} = Ax + B_1w + B_2u$$
$$v = C_1Ax + C_1B_1w + C_1B_2u$$

3. Substitution of this v into z leads to

$$z = (C_2 - C_1A)x - C_1B_1w - C_1B_2u.$$

4. Norm of matrix uncertainty Δ



Descriptor form

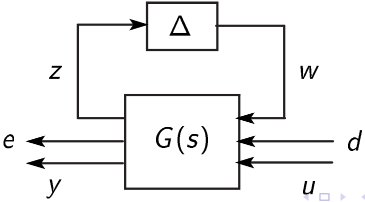
1. General model for norm-bounded parametric systems

$$G \begin{cases} \dot{x} = Ax + B_1 w + B_2 d + B_3 u \\ z = C_1 x + D_{11} w + D_{12} d + D_{13} u \\ e = C_2 x + D_{21} w + D_{22} d + D_{23} u \\ y = C_3 x + D_{31} w + D_{32} d \end{cases} \tag{23}$$

$$w = \Delta z, \quad \|\Delta\|_2 \leq 1. \tag{24}$$

2. Parameter uncertainty can be time-varying

$$\Delta(t) \in \mathbb{R}^{p \times q}, \quad \|\Delta(t)\|_2 \leq 1 \; \forall t \geq 0. \tag{25}$$

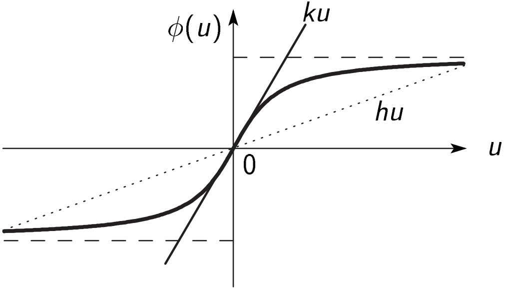


Phase Information of Uncertainty

1. How to capture static nonlinearity such as dead-zone, saturation?
2. Example: saturation function $\phi(\cdot)$ can be enclosed by two lines with slopes of k and 0 , and described by

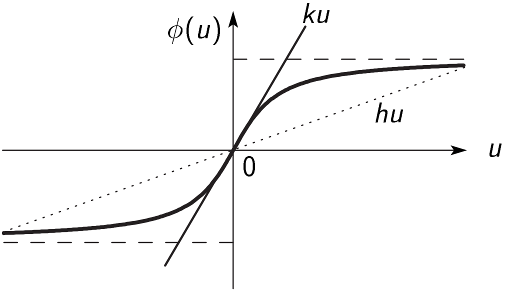
$$0 \leq u\phi(u) \leq ku^2 \quad \forall u. \tag{26}$$

3. Relationship between the magnetic flux and the current in an electromagnet is exactly such a saturation.

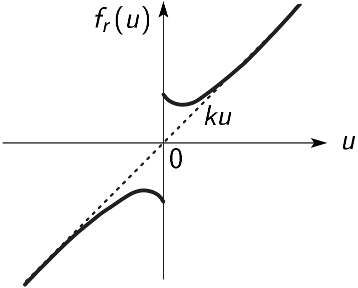


Phase Information of Uncertainty

1. More important character: $\phi(\cdot)$ is located in the 1st and the 3rd quadrants, namely its slope changes between 0 and k.
2. Phase angle of $\phi(\cdot)$ is zero while its gain changes in the interval $[0, k]$.
3. Particularly useful is the information on the phase angle.
4. Stribeck friction also has such a feature.



(a) Saturation



(b) Stribeck friction

Figure: Examples of passive nonlinearity

Phase Information of Uncertainty

Flexible structure: torque input, velocity output

$$P(s) = \frac{k_0 s}{s^2 + 2\zeta_0\omega_0 s + \omega_0^2} + \frac{k_1 s}{s^2 + 2\zeta_1\omega_1 s + \omega_1^2} + \cdots + \frac{k_n s}{s^2 + 2\zeta_n\omega_n s + \omega_n^2}$$
$$k_i > 0 \text{ in} - \text{phase} \tag{27}$$

1. 1st resonant mode $\frac{k_0 s}{s^2+2\zeta_0\omega_0 s+\omega_0^2}$ can be identified, but the 2nd and higher resonant modes can hardly be identified correctly.
2. Frequency response of a mode $\frac{s}{s^2+2\zeta_i\omega_i s+\omega_i^2}$:

$$\frac{j\omega}{\omega_i^2 - \omega^2 + j2\zeta_i\omega_i\omega} = \frac{2\zeta_i\omega_i\omega^2 + j\omega(\omega_i^2 - \omega^2)}{(\omega_i^2 - \omega^2)^2 + (2\zeta_i\omega_i\omega)^2}$$

3. Real part is nonnegative when $\zeta_i \geq 0$, thus a PR function.

Phase Information of Uncertainty

1. Therefore, the sum of high order resonant modes is also PR:

$$\frac{k_1 s}{s^2 + 2\zeta_1\omega_1 s + \omega_1^2} + \cdots + \frac{k_n s}{s^2 + 2\zeta_n\omega_n s + \omega_n^2}$$

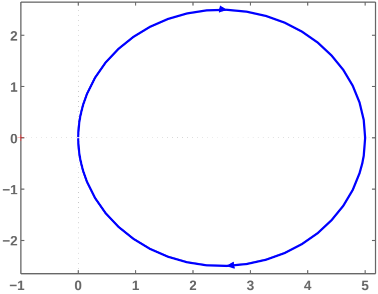


Figure: Nyquist diagram of a positive real transfer function

LPV Model

1. When the operating range of nonlinear systems is large, a linear approximation can no longer be trusted.
2. Meanwhile, the need is very strong to use techniques similar to linear control in the control of nonlinear systems.
3. How to transform a nonlinear system to a quasi-linear (namely in a linear form) system?
4. *LPV* (linear parameter varying) model

$$\dot{x} = A(p(t))x + B(p(t))u \tag{28}$$

$$y = C(p(t))x. \tag{29}$$

   $p(t)$ is a time-varying parameter vector and each matrix is an affine function of $p(t)$.
5. $p(t) = [p_1(t)\ p_2(t)]$ case

   $A(p(t)) = A_0 + p_1(t)A_1 + p_2(t)A_2,\ B(p(t)) = B_0 + p_1(t)B_1 + p_2(t)B_2$

   $C(p(t)) = C_0 + p_1(t)C_1 + p_2(t)C_2$

From Nonlinear System to LPV Model

$$\dot{\delta} = \omega - \omega_0 \tag{30}$$

$$\dot{\omega} = \frac{\omega_0}{M} P_M - \frac{\omega_0}{M} P_e - \frac{D}{M}(\omega - \omega_0) \tag{31}$$

$$\dot{E}'_q = -\frac{1}{T'_d} E'_q + \frac{x_d - x'_d}{T_{d0} x'_{d\Sigma}} V_s \cos\delta + \frac{1}{T_{d0}} V_f. \tag{32}$$

$$P_e = \frac{E'_q V_s}{x'_{d\Sigma}} \sin\delta - \frac{V_s^2}{2} \frac{x_d - x'_d}{x'_{d\Sigma} x_{d\Sigma}} \sin 2\delta. \tag{33}$$

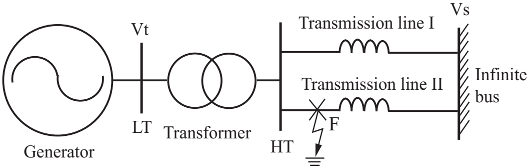


Figure: Single-machine infinite-bus power system

From Nonlinear System to LPV Model

1. Transient stability: stability in face of short-circuit fault
2. Short-circuit fault makes the active power drop instantly, thus causing a steep acceleration of the rotor (see Eq.(31)).
3. If no excitation control is activated quickly, the generator will lose synchronism.
4. In the fault phase, all states deviate from the equilibrium significantly so that the linear approximation fails.
5. Equilibrium of power system: $(\delta_0,\ \omega_0,\ E'_{q0},\ V_{f0})$.
6. Goal of control: restore the deviated states back to the equilibrium, i.e. the stabilization of error states

$$x_1 = \delta - \delta_0,\ x_2 = \omega - \omega_0,\ x_3 = E'_q - E'_{q0},\ u = V_f - V_{f0}.$$

From Nonlinear System to LPV Model

1. State equation

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= d_1 \sin\delta \cdot x_3 + d_1 E'_{q0}(\sin\delta - \sin\delta_0) + d_2 x_2 \\ \dot{x}_3 &= d_3 x_3 + d_4(\cos\delta - \cos\delta_0) + d_5 x_4. \end{aligned} \tag{34}$$

2. Since

$$\sin\delta - \sin\delta_0 = \frac{\sin\delta - \sin\delta_0}{\delta - \delta_0} x_1, \quad \cos\delta - \cos\delta_0 = \frac{\cos\delta - \cos\delta_0}{\delta - \delta_0} x_1$$

3. Time-varying parameters defined as functions of rotor angle δ:

$$p_1(\delta) = \frac{\sin\delta - \sin\delta_0}{\delta - \delta_0}, \quad p_2(\delta) = \sin\delta, \quad p_3(\delta) = \frac{\cos\delta - \cos\delta_0}{\delta - \delta_0} \tag{35}$$

From Nonlinear System to LPV Model

1. LPV model

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= d_1 p_2(\delta) x_3 + d_1 E'_{q0} p_1(\delta) x_1 + d_2 x_2 \\
\dot{x}_3 &= d_3 x_3 + d_4 p_3(\delta) x_1 + d_5 x_4
\end{aligned}
\tag{36}
$$

2. As long as the rotor angle $\delta(t)$ is measured, the time-varying parameter vector $p(t) = [p_1 \;\; p_2 \;\; p_3]^T$ can always be calculated online.
3. Vector form

$$
\dot{x} = A(p)x + Bu \tag{37}
$$

$$
A(p) = \begin{bmatrix} 0 & 1 & 0 \\ d_1 E'_{q0} p_1(\delta) & d_2 & d_1 p_2(\delta) \\ d_4 p_3(\delta) & 0 & d_3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ d_5 \end{bmatrix}
$$