Chapter 16

$\mathcal{H}_\infty$ Control

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$\mathcal{H}_\infty$ Norm of Transfer Function

1. Transfer matrix

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \tag{1}$$

2. $\mathcal{H}_\infty$ norm: maximal amplitude of freq. response

$$\|G\|_\infty := \max_\omega |G(j\omega)| \qquad \text{SISO} \tag{2}$$

$$\|G\|_\infty := \max_\omega \sigma_{\max}(G(j\omega)) \qquad \text{MIMO} \tag{3}$$

Relation with Input and Output: 1

$$\|G\|_\infty = \sup_{\|u\|_2 \neq 0} \frac{\|y\|_2}{\|u\|_2}. \tag{4}$$

1. $\|y\|_2/\|u\|_2$: ratio of input and output energies. Its supremum for all energy-bounded input $u(t)$ is the $\mathcal{H}_\infty$ norm. $\|G\|_\infty$.
2. To lower the output response $y(t)$ to a energy-bounded disturbance $u(t)$, we need
$$\|G\|_\infty \to 0.$$
3. To make the input-output ratio less than a given value $\gamma >$, it is sufficient to guarantee
$$\|G\|_\infty < \gamma.$$

Relation with Input and Output: 2

1. Instead of energy bounded, a disturbance is persistent whose energy is unbounded. New viewpoint needed.
2. SISO system: maximum amplitude of system's frequency response to unit impulse input

$$\|G\|_\infty = \sup_\omega |G(j\omega)|$$

3. MIMO system

$$\|G\|_\infty = \sup_{\substack{u\in\mathbb{C}^m \\ \|u\|\le 1}} \|Gu\|_\infty , \quad \|Gu\|_\infty = \sup_\omega \|G(j\omega)u\|_2 . \tag{5}$$

Complex space $\mathbb{C}^m$ is a space of impulse vector signals containing time-delay. So, $\|G\|_\infty$ is the maximum amplitude of all the frequency responses w.r.t. unit impulse vectors whose elements are imposed at arbitrary instants.

Weighting Function vs Disturbance

1. Disturbance has dynamics $W(s)$. Then $\hat{y}(s) = G(s)W(s)$.
2. Suppression of the disturbance response

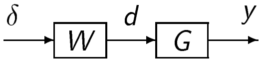
$$\|\hat{y}\|_\infty \leq \|GW\|_\infty < \gamma \tag{6}$$

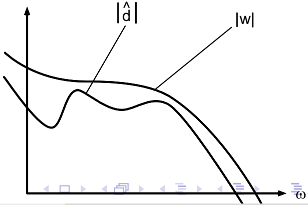
3. Even when only an upper bound $|W(j\omega)|$ is known,

$$|\hat{d}(j\omega)| \leq |W(j\omega)| \quad \forall\ \omega,$$

disturbance is still suppressed if $\|GW\|_\infty$ is minimized because

$$\left\| G\hat{d} \right\|_\infty \leq \|GW\|_\infty .$$





An example

Example 1

Ref tracking: plant $P(s) = 1/s$, controller $K(s) = k$, ref. input $r(t) = 1(t)$. Seek a gain k s.t. the tracking error $e(t)$ satisfies the performance specification $\sup_\omega |\hat{e}(j\omega)| \leq 0.1$.

(Solution) $r(t)$ is the unit impulse response of $W(s) = 1/s$. Hence, $e(t)$ becomes the unit impulse response of the weighted transfer function WS. Therefore, $\sup_\omega |\hat{e}(j\omega)| = \|WS\|_\infty$ in which

$$S(s) = \frac{1}{1+PK} = \frac{s}{s+k}$$

$k > 0$ is necessary for the internal stability, Then,

$$\|WS\|_\infty = \left\| \frac{1}{s+k} \right\|_\infty = \frac{1}{k} \leq 0.1 \ \Rightarrow \ k \geq 10.$$

$\mathcal{H}_\infty$ control problem

1. For any given $\gamma > 0$, design a controller satisfying $\|H_{zw}\|_\infty < \gamma$.
2. (A1): (A, B_2) is stabilizable and (C_2, A) is detectable.

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right] \tag{7}$$

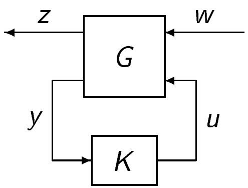


Figure: Generalized feedback system

Solution 1: Variable Elimination

$$N_Y = [C_2 \;\; D_{21}]_\perp, \;\; N_X = [B_2^T \;\; D_{12}^T]_\perp.$$

Theorem 1

Assume (A1). The $\mathcal{H}_\infty$ problem $\|H_{zw}\|_\infty < \gamma$ has a solution iff $\exists X > 0, Y > 0$ satisfying

$$\begin{bmatrix} N_X^T & 0 \\ 0 & I_{n_w} \end{bmatrix} \begin{bmatrix} AX + XA^T & XC_1^T & B_1 \\ C_1 X & -\gamma I & D_{11} \\ B_1^T & D_{11}^T & -\gamma I \end{bmatrix} \begin{bmatrix} N_X & 0 \\ 0 & I_{n_w} \end{bmatrix} < 0 \qquad (8)$$

$$\begin{bmatrix} N_Y^T & 0 \\ 0 & I_{n_z} \end{bmatrix} \begin{bmatrix} YA + A^T Y & YB_1 & C_1^T \\ B_1^T Y & -\gamma I & D_{11}^T \\ C_1 & D_{11} & -\gamma I \end{bmatrix} \begin{bmatrix} N_Y & 0 \\ 0 & I_{n_z} \end{bmatrix} < 0 \qquad (9)$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0, \quad \text{rank} \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \leq n + n_K. \qquad (10)$$

(Proof) CLS: $H_{zw}(s) = (A_c, B_c, C_c, D_c)$.
According to the bounded-real lemma, $\mathcal{H}_\infty$ problem is solvable iff $\exists P > 0$ satisfying

$$\begin{bmatrix} A_c^T P + PA_c & PB_c & C_c^T \\ B_c^T P & -\gamma I & D_c^T \\ C_c & D_c & -\gamma I \end{bmatrix} < 0. \tag{11}$$

OR equivalently

$$Q + E^T \mathcal{K} F + F^T \mathcal{K}^T E < 0 \tag{12}$$

$$\begin{bmatrix} Q & E^T \\ F & \end{bmatrix} = \left[\begin{array}{ccc:c} \overline{A}^T P + P\overline{A} & P\overline{B}_1 & \overline{C}_1^T & P\overline{B}_2 \\ \overline{B}_1^T P & -\gamma I & \overline{D}_{11}^T & 0 \\ \overline{C}_1 & \overline{D}_{11} & -\gamma I & \overline{D}_{12} \\ \hdashline \overline{C}_2 & \overline{D}_{21} & 0 & \end{array}\right], \quad \mathcal{K} = \begin{bmatrix} D_K & C_K \\ B_K & A_K \end{bmatrix}.$$

Owing to Theorem 3.1, (12) is equivalent to

$$E_\perp^T Q E_\perp < 0, \quad F_\perp^T Q F_\perp < 0. \tag{13}$$

Decomposition of P

$$P = \begin{bmatrix} Y & * \\ * & * \end{bmatrix}, \; P^{-1} = \begin{bmatrix} X & * \\ * & * \end{bmatrix}.$$

Then, conditions (8), (9) are derived from (13). Condition (10) is obtained from the positive definiteness of matrix P (Lemma 3.1). ∇

Solution 2: Variable Change

Factorization of matrix $P > 0$

$$P\Pi_1 = \Pi_2, \quad \Pi_1 = \begin{bmatrix} X & I \\ M^T & 0 \end{bmatrix}, \ \Pi_2 = \begin{bmatrix} I & Y \\ 0 & N^T \end{bmatrix}.$$

Variable change

$$\begin{aligned} \mathbb{A} &= NA_K M^T + NB_K C_2 X + YB_2 C_K M^T + Y(A + B_2 D_K C_2)X \\ \mathbb{B} &= NB_K + YB_2 D_K, \ \mathbb{C} = C_K M^T + D_K C_2 X, \ \mathbb{D} = D_K \end{aligned} \tag{14}$$

Notation $\mathrm{He}(A) = A + A^T$

Solution 2: Variable Change

Solvability condition

$$\mathrm{He}\begin{bmatrix} AX + B_2\mathbb{C} & A + B_2\mathbb{D}C_2 & B_1 + B_2\mathbb{D}D_{21} & 0 \\ \mathbb{A} & YA + \mathbb{B}C_2 & YB_1 + \mathbb{B}D_{21} & 0 \\ 0 & 0 & -\frac{\gamma}{2}I & 0 \\ C_1X + D_{12}\mathbb{C} & C_1 + D_{12}\mathbb{D}C_2 & D_{11} + D_{12}\mathbb{D}D_{21} & -\frac{\gamma}{2}I \end{bmatrix} < 0 \quad (15)$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0. \quad (16)$$

Controller $K(s) = (A_K, B_K, C_K, D_K)$

$$D_K = \mathbb{D},\;\; C_K = (\mathbb{C} - D_K C_2 X)(M^{-1})^T,\;\; B_K = N^{-1}(\mathbb{B} - YB_2D_K)$$
$$A_K = N^{-1}(\mathbb{A} - NB_K C_2 X - YB_2 C_K M^T - Y(A + B_2 D_K C_2)X)(M^{-1})^T. \quad (17)$$

Selection of Generalized Plant

1. Consideration of Disturbance Control
   - Single out the major disturbance and put its output response into the performance output.
   - Examine the frequency response of disturbance and use it the weighting function.
2. Consideration of Model Uncertainty
3. Consideration of Input Constraint
   - Always put the input into the performance output

Selection of Weighting Function

1. Weighting Function of Dynamic Uncertainty
   - Use a tight but low-order upper bound
2. Weighting Function of Input
   - High-pass transfer function
   - Low gain within the control bandwidth
   - High gain beyond the control bandwidth
3. Weighting Function of Performance
   - Use models of ref. input and disturbance, usually low-pass;
   - Raise the gain as high as possible

IEEJ HDD benchmark

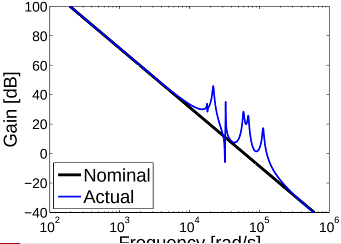
1. Physical model of HDD

$$\tilde{P}(s) = \frac{K_p}{s^2} + \frac{A_1}{s^2 + 2\zeta_1\omega_1 s + \omega_1^2} + \frac{A_1}{s^2 + 2\zeta_1\omega_2 s + \omega_2^2} + \cdots \qquad (18)$$

2. Obtained via finite element method and modal analysis
3. High order resonant modes vary with manufacturing error, hundreds of thousands of HDDs controlled by the same controller.
4. Control design carried out based on rigid body model $P(s) = K_p/s^2$

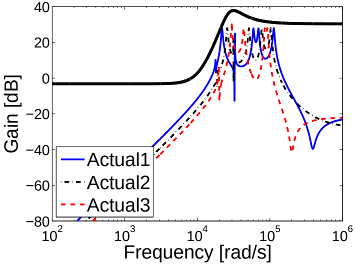
IEEJ HDD benchmark

i	f_i (Hz)	ζ_i	A_i
1	4100 ($\pm$15%)	0.02	-1.0
2	8200 ($\pm$15%)	0.02	1.0
3	12300 ($\pm$10%)	0.02	-1.0
4	16400 ($\pm$10%)	0.02	1.0
5	3000 ($\pm$5%)	0.005	0.01 ($-200\% \sim 0\%$)
6	5000 ($\pm$5%)	0.001	0.03 ($-200\% \sim 0\%$)
K_p		3.744×10^9	



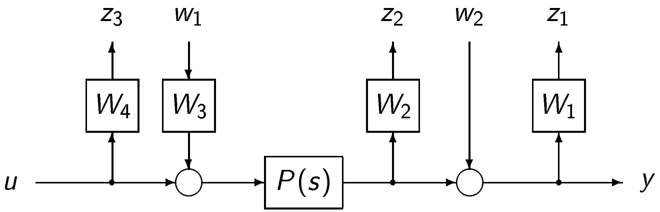
IEEJ HDD benchmark

1. High freq resonant modes modeled as a multiplicative uncertainty
$$P(s) = P_0(1 + \Delta W), \quad P_0(s) = \frac{k}{s^2}, \quad \|\Delta\|_\infty \leq 1.$$
2. Draw the relative error $\left|1 - \frac{P(j\omega)}{P_0(j\omega)}\right|$ in a Bode plot.
3. Determine a minimum phase weighting function s.t. the gain of its freq response covers the relative errors.



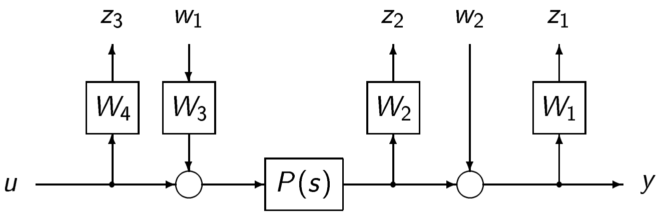
Case Study: $\mathcal{H}_\infty$ control of HDD

1. Head positioning in face of wind disturbance
2. Wind disturbance: step signal
3. w_1 and z_1: input and output used to penalize the disturbance response
4. w_2 and z_2: output and input of multiplicative uncertainty
5. z_3: performance output used to penalize the control input u



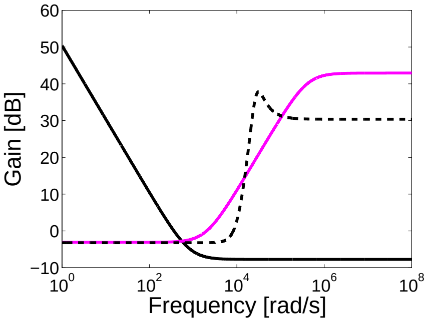
Case Study: $\mathcal{H}_\infty$ control of HDD

1. W_1: dynamics of the disturbance
2. W_2: gain of the multiplicative uncertainty
3. W_3: parameter mainly used to tune the response speed
4. W_4: weighting function used to adjust the control input



Weighting functions

1. Gain of uncertainty weight W_2 rises sharply around $\omega = 2 \times 10^4$ rad/s, no effective control possible above it.
2. Disturbance weight W_1 and input weight W_4 should intersect in the vicinity of this frequency.
3. Wind disturbance model W_1 is an integrator, its gain should be as high as possible.

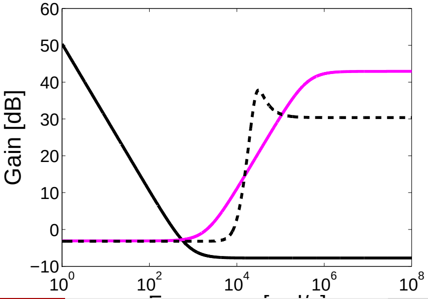


Weighting functions

$$W_1(s) = \frac{s + 8.1 \times 10^2}{s + 1.0 \times 10^{-6}} \times 4.1 \times 10^{-4}$$

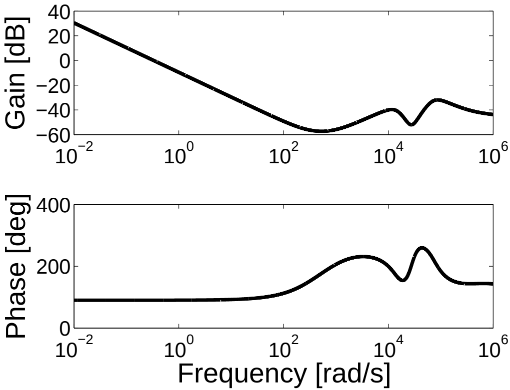
$$W_2(s) = \left(\frac{s^2 + 1.4 \times 10^4 s + 1.1 \times 10^8}{s^2 + 1.9 \times 10^4 s + 7.6 \times 10^8}\right)^2 \times 33$$

$$W_3 = 1.0 \times 10^{-3}, \;\; W_4(s) = \frac{s + 2.0 \times 10^3}{s + 4.0 \times 10^5} \times 1.4 \times 10^2.$$



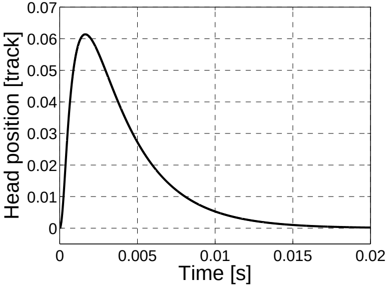
$\mathcal{H}_\infty$ controller

1. Notch at the peak freq. of resonant modes, obtained automatically.
2. Contains an integrator
3. Phase lead

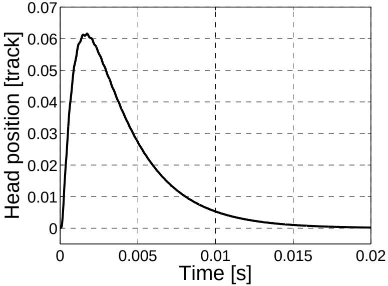


Output Response

1. No noticeable difference in outputs, almost the same output response have been achieved.



(a) Nominal output response

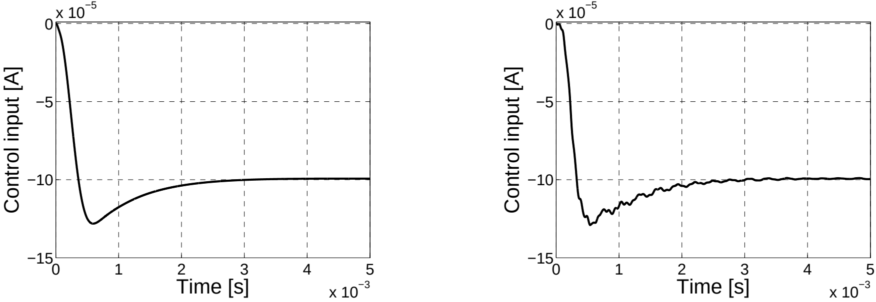


(b) Actual output response

Figure: Step disturbance response (output)

Input Response

1. Input of the actual system is much more oscillatory.



(a) Nominal output response (b) Actual output response

Figure: Step disturbance response (input)

Scaled $\mathcal{H}_\infty$ Control

1. $\mathcal{H}_\infty$ control problems with a constant scaling matrix $L > 0$

$$\|L^{1/2} H_{zw} L^{-1/2}\|_\infty < \gamma. \tag{19}$$

2. CLS $H_{zw}(s) = (A_c, B_c, C_c, D_c)$ satisfies (19) iff $\exists P > 0,\ L > 0$ satisfying (bounded-real lemma)

$$\begin{bmatrix} A_c^T P + PA_c & PB_c & C_c^T \\ B_c^T P & -\gamma L & D_c^T \\ C_c & D_c & -\gamma L^{-1} \end{bmatrix} < 0. \tag{20}$$

Solution

Theorem 2

Assume (A1). The scaled $\mathcal{H}_\infty$ problem has a solution iff there exist matrices $X > 0$, $Y > 0$ and L, J satisfying

$$\begin{bmatrix} N_X^T & 0 \\ 0 & I_{n_w} \end{bmatrix} \begin{bmatrix} AX + XA^T & XC_1^T & B_1 \\ C_1X & -\gamma J & D_{11} \\ B_1^T & D_{11}^T & -\gamma L \end{bmatrix} \begin{bmatrix} N_X & 0 \\ 0 & I_{n_w} \end{bmatrix} < 0 \tag{21}$$

$$\begin{bmatrix} N_Y^T & 0 \\ 0 & I_{n_z} \end{bmatrix} \begin{bmatrix} YA + A^TY & YB_1 & C_1^T \\ B_1^TY & -\gamma L & D_{11}^T \\ C_1 & D_{11} & -\gamma J \end{bmatrix} \begin{bmatrix} N_Y & 0 \\ 0 & I_{n_z} \end{bmatrix} < 0 \tag{22}$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0 \tag{23}$$

$$LJ = I. \tag{24}$$

K-L iteration mathod

1. Condition $LJ = I$ is not convex, cannot be solved by using LMI approach directly.
2. Need to use the *K-L iteration method*.

Step 1 Let $L = I$.

Step 2 Compute a controller $K(s)$ such that $\|L^{1/2}H_{zw}L^{-1/2}\|_\infty$ is minimized and denote the minimal norm by γ_K.

Step 3 Fixing the controller $K(s)$, find scaling matrix $L > 0$ such that $\|L^{1/2}H_{zw}L^{-1/2}\|_\infty$ is minimized and denote the minimal norm by γ_L.

Step 4 If $\gamma_K - \gamma_L$ is less than a specified value, end the design and output the controller $K(s)$ obtained in Step 2; Otherwise, return to Step 2.

K-L iteration mathod

1. L is known in Step 2, so we can compute γ_K and $P > 0$ by solving a GEVP:
$$\min \gamma \text{ subject to (21), (22), (23)}$$
2. Controller $K(s)$ is obtained by solving LMI
$$Q + E^T \mathcal{K} F + F^T \mathcal{K}^T E < 0 \tag{25}$$
$$\begin{bmatrix} Q & E^T \\ F & \end{bmatrix} = \left[\begin{array}{ccc|c} \overline{A}^T P + P\overline{A} & P\overline{B}_1 & \overline{C}_1^T & P\overline{B}_2 \\ \overline{B}_1^T P & -\gamma_K L & \overline{D}_{11}^T & 0 \\ \overline{C}_1 & \overline{D}_{11} & -\gamma_K J & \overline{D}_{12} \\ \hline \overline{C}_2 & \overline{D}_{21} & 0 & \end{array}\right]. \tag{26}$$
3. Optimization problem in Step 3 can be solved by solving the GEVP:
$$\min \gamma \text{ subject to}$$
$$\begin{bmatrix} A_c^T P + PA_c & PB_c & C_c^T L \\ B_c^T P & -\gamma L & D_c^T L \\ LC_c & LD_c & -\gamma L \end{bmatrix} < 0, \quad P > 0, \quad L > 0 \tag{27}$$

Project

- Repeat the design of head positioning control of HDD using the scaled $\mathcal{H}_\infty$ control method.
- Requirements
    1. Tune the weighting functions of performance output so as to achieve the best possible solution.
    2. Show the Bode plots of controller, open-loop systems w.r.t. the nominal model and true plants at all vertices of parameter vectors.
    3. Show the Bode plots of closed-loop systems from wind disturbance to head position w.r.t. the nominal model and true plants at all vertices of parameter vectors.
    4. Show the time response of head position and input w.r.t. the nominal model and true plants at all vertices of parameter vectors.