Chapter 2

Basics of linear algebra

Table of contents

Vector norm

- How to measure the size of a vector?
- Distance in 3D Euclidean space: from a point $P(x, y, z)$ to the origin

$$d(P) = \sqrt{x^2 + y^2 + z^2}. \tag{1}$$

- Notion of norm

$$\|u\| = d(P), \quad P(x, y, z) \sim u = [x \ y \ z]^T.$$

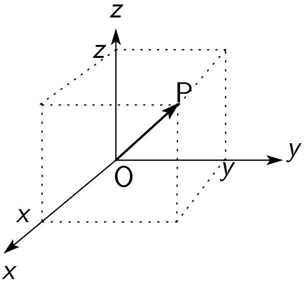


Figure: Distance in Euclidean space

Vector norm

- Property of Euclidean distance
  1. $||u|| \geq 0$ (positivity)
  2. $||u|| = 0$ iff $u \equiv 0$ (positive definiteness)
  3. $||\alpha u|| = |\alpha|\,||u||$ for any scalar $\alpha \in \mathbb{R}$ (homogeneity)
  4. $||u + v|| \leq ||u|| + ||v||$ for any vectors u, v (triangle inequality)

(1): trivial.
(2): $||u|| = \sqrt{x^2 + y^2 + z^2} = 0 \Leftrightarrow x = y = z = 0 \Leftrightarrow u = 0$
(3): $||\alpha u|| = \sqrt{(\alpha x)^2 + (\alpha y)^2 + (\alpha z)^2} = |\alpha|\,||u||$
(4): Proved by Cauchy-Schwarz inequality

$$2(xy + yz + zx) \leq x^2 + y^2 + z^2.$$

Vector norm

- Generalization: *scalar real-valued function* defined in any vector space (as well as matrix space, function space) is called a norm of the corresponding space if it satisfies all the properties below, and is used to measure the size of vector (matrix, function).
  1. $\|u\| \geq 0$
  2. $\|u\| = 0$ iff $u \equiv 0$
  3. $\|\alpha u\| = |\alpha| \, \|u\|$ for any scalar $\alpha \in \mathbb{F}$
  4. $\|u + v\| \leq \|u\| + \|v\|$ for any vectors (matrices, functions) u, v

Examples:

1-norm $\|u\|_1 = \sum_{i=1}^{n} |u_i|$

2-norm $\|u\|_2 = \sqrt{\sum_{i=1}^{n} |u_i|^2}$

Infinity-norm $\|u\|_\infty = \max_{1 \leq i \leq n} |u_i|$

An example

Example 1

Prove that the function $f(u) = \sum_{i=1}^{n} |u_i|$ is a norm.

(Proof) Obviously, $f(u) \geq 0$. Secondly

$$f(u) = 0 \;\Leftrightarrow\; |u_i| = 0 \;\forall i \;\Leftrightarrow\; u_i = 0 \;\forall i \;\Leftrightarrow\; u = 0$$

holds. It is also easy to see that

$$f(\alpha u) = \sum_{i=1}^{n} |\alpha u_i| = |\alpha| \sum_{i=1}^{n} |u_i| = |\alpha| f(u).$$

Further,

$$f(u+v) = \sum_{i=1}^{n} |u_i + v_i| \leq \sum_{i=1}^{n} (|u_i| + |v_i|) = f(u) + f(v)$$

is true because the triangle inequality $|u_i + v_i| \leq |u_i| + |v_i|$ holds for scalars. So, this $f(u)$ is indeed a norm.

Inner Product of Vector

- How to describe the direction relation between vectors, i.e. the angle between them?
- 2D Euclidean space $\mathbb{R}^2$: $u_i = [x_i \ y_i]^T \ (i = 1, 2)$

$$\|u_1 - u_2\|_2^2 = \|u_1\|_2^2 + \|u_2\|_2^2 - 2\|u_1\|_2 \|u_2\|_2 \cos\theta \tag{2}$$

$$\Rightarrow \cos\theta = \frac{x_1 x_2 + y_1 y_2}{\|u_1\|_2 \|u_2\|_2} = \frac{u_1^T u_2}{\|u_1\|_2 \|u_2\|_2}. \tag{3}$$

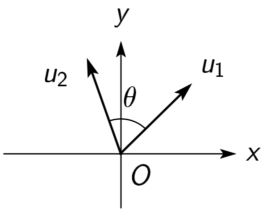


Figure: Inner product and angel

Inner Product of Vector

- $u_1^T u_2$ is a function mapping two vectors into a scalar, called *inner product* and denoted by

$$\langle u_1, u_2 \rangle := u_1^T u_2. \tag{4}$$

- Then, we have

$$\cos\theta = \frac{\langle u_1, u_2 \rangle}{\|u_1\|_2 \|u_2\|_2}, \qquad \theta \in [0, \pi]. \tag{5}$$

- Therefore, inner product and angle have a one-to-one relationship.
- In vector spaces with higher dimensions as well as matrix and function spaces to be described later on, the angle cannot be drawn. So, it is necessary to use the inner product to define the angle between the elements of each space.

Example

Example 2

Given vectors

$$u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ v = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \ w = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Let the angle between u, v be ϕ and the angle between u, w be θ. Calculation based on inner product yields

$$\cos\phi = \frac{u^T v}{\|u\|_2 \, \|v\|_2} = 0 \ \Rightarrow \ \phi = 90^\circ,$$
$$\cos\theta = \frac{u^T w}{\|u\|_2 \, \|w\|_2} = \frac{1}{\sqrt{2}} \ \Rightarrow \ \theta = 45^\circ.$$

We can verify the correctness of the calculation by drawing a figure.

Generalizations

- Inner product between real-valued vectors $u, v \in \mathbb{R}^n$

$$\langle u, v \rangle := u^T v. \tag{6}$$

- Inner product between complex-valued vectors $u, v \in \mathbb{C}^n$

$$\langle u, v \rangle := u^* v \tag{7}$$

- Angle between two vectors u, v

$$\cos\theta = \frac{\Re(\langle u, v \rangle)}{\|u\|_2 \, \|v\|_2}, \quad \theta \in [0, \pi]. \tag{8}$$

- Reason for its definition
  Complex-valued vector u and real-valued vector $\begin{bmatrix} \Re(u) \\ \Im(u) \end{bmatrix}$ are one-to-one,

$$\Re(\langle u, v \rangle) = [\Re(u) \; \Im(u)] \begin{bmatrix} \Re(v) \\ \Im(v) \end{bmatrix}.$$

Properties of inner product

1. $\langle x, \alpha y + \beta z\rangle = \alpha \langle x, y\rangle + \beta \langle x, z\rangle$ holds for any scalars $\alpha, \beta \in \mathbb{F}$.
2. $\langle x, y\rangle = \overline{\langle y, x\rangle}$.
3. $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ iff $x = 0$.
4. *induced norm* $\sqrt{\langle u, u\rangle} = \|u\|_2$

Theorem 1

For any $u, v \in \mathbb{F}^n$, the following statements are true.

1. $|\langle u, v\rangle| \leq \|u\|_2 \|v\|_2$ *(Cauchy-Schwarz inequality). The equality hods only when $u = \alpha v$ (α is a constant), $u = 0$ or $v = 0$.*
2. $\|u + v\|_2^2 + \|u - v\|_2^2 = 2\|u\|_2^2 + 2\|v\|_2^2$ *(Parallelogram law)*
3. $\|u + v\|_2^2 = \|u\|_2^2 + \|v\|_2^2$ *when $u \perp v$ (Pythagoras law)*

Quadratic Form and Energy Function

1. $ax_1^2 + 2bx_1x_2 + cx_2^2$ of vector $x = [x_1 \;\; x_2]^T$ is called a quadratic form, usually related with energy of a physical system
2. Examples: kinetic energy $mv^2/2$ of a mass m, rotational energy $J\omega^2/2$ of a rigid body with inertia J
3. System stability, or control performance are closely related to energy, quadratic form often encountered in systems analysis and design.
4. nD case

$$
\begin{aligned}
V(x) =& \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}x_ix_j \\
=&(b_{11}x_1^2 + b_{12}x_1x_2 + \cdots + b_{1n}x_1x_n) + \cdots \\
&+ (b_{n1}x_nx_1 + b_{n2}x_nx_2 + \cdots + b_{nn}x_n^2)
\end{aligned} \tag{9}
$$

Quadratic Form and Energy Function

Using $x_i x_j = x_j x_i$ and setting

$$a_{ii} = b_{ii}, \ a_{ij} = a_{ji} = \frac{b_{ij} + b_{ji}}{2}, \ i \neq j, \tag{10}$$

we can always write $V(x)$ as

$$V(x) = x^T A x, \quad A = (a_{ij}) = A^T \tag{11}$$

For example,

$$a x_1^2 + 2b x_1 x_2 + c x_2^2 = [x_1 \ x_2] \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

1. Complex case: $x \in \mathbb{C}^n$

$$V(x) = x^* A x \in \mathbb{R}. \tag{12}$$

Positive Definite/Positive Semi-definite Matrices

- Energy is always positive. So the quadratic form describing energy should also be positive.
- Positive definite function: $V(x) = x^*Ax > 0$ for any $x \neq 0$
- Positive semi-definite function: $V(x) = x^*Ax \geq 0$ for any $x \neq 0$
- Positive definite matrix: Hermitian matrix $A = A^*$ satisfies $x^*Ax > 0$ for any $x \neq 0$, denoted by $A > 0$.
- Positive semi-definite matrix: $x^*Ax \geq 0$ for any $x \neq 0$, denoted by $A \geq 0$.
- Example: $B^*B \geq 0$ for matrix B

$$\because \; x^*B^*Bx = \|Bx\|_2^2 \geq 0 \quad \forall\ x$$

Positive Definite/Positive Semi-definite Matrices

Theorem 2

When $A \in \mathbb{F}^{n\times n}$ is Hermitian, the following statements hold.

1. $A \geq 0$ *iff its eigenvalues are all nonnegative.*
2. $A > 0$ *iff its eigenvalues are all positive.*
3. *When $A \geq 0$, there exists $B \in \mathbb{F}^{n\times r}$ such that A is decomposed as $A = BB^*$ where $r \geq \mathrm{rank}(A)$.*

Schur's Lemma

Lemma 1 (Schur's Lemma)

Partition Hermitian matrix $X = X^*$ as

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{bmatrix}$$

in which X_{11}, X_{22} are square. Then, the following statements are true.

1. $X > 0$ *iff one of the following conditions is satisfied.*
    1. $X_{22} > 0$, $X_{11} - X_{12}X_{22}^{-1}X_{12}^* > 0$.
    2. $X_{11} > 0$, $X_{22} - X_{12}^*X_{11}^{-1}X_{12} > 0$.
2. $X \geq 0$ *iff one of the following conditions holds.*
    1. $X_{22} \geq 0$, $\mathrm{Ker}X_{22} \subset \mathrm{Ker}X_{12}$, $X_{11} - X_{12}X_{22}^{\dagger}X_{12}^* \geq 0$.
    2. $X_{11} \geq 0$, $\mathrm{Ker}X_{11} \subset \mathrm{Ker}X_{12}^*$, $X_{22} - X_{12}^*X_{11}^{\dagger}X_{12} \geq 0$.

Matrix Norm

- Vector mapped by matrix

$$Au \longleftarrow \boxed{A} \longleftarrow u$$

- Matrix can be regarded as an amplifier, and vector as a signal.
- Matrix norm can be regarded as the amplification rate of signal, defined by the ratio of input and output vector norms.
- Ratio of input and output vector norms is not a constant, varies with the direction of input vector.

Example 3

Mapping $u_1 = [1\ \ 0]^T$, $u_2 = [0\ \ 1]^T$ *and* $u_3 = [1\ \ 1]^T/\sqrt{2}$ *by* $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, *we get output vectors* $y_1 = [1\ \ 3]^T$, $y_2 = [2\ \ 4]^T$ *and* $y_3 = [3\ \ 7]^T/\sqrt{2}$ *resp. Therefore, the 2-norm ratio of input and output are resp* $\sqrt{10},\ 2\sqrt{5},\ \sqrt{29}$.

Matrix Norm

$$\|A\|_1 := \sup_{u \neq 0} \frac{\|Au\|_1}{\|u\|_1} \tag{13}$$

$$\|A\|_2 := \sup_{u \neq 0} \frac{\|Au\|_2}{\|u\|_2} \tag{14}$$

$$\|A\|_\infty := \sup_{u \neq 0} \frac{\|Au\|_\infty}{\|u\|_\infty} \tag{15}$$

1-norm $\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{m} |a_{ij}|$ (column sum)

2-norm $\|A\|_2 = \sqrt{\lambda_{\max}(A^* A)}$

Infinity-norm $\|A\|_\infty = \max_{1 \leq i \leq m} \sum_{j=1}^{n} |a_{ij}|$ (row sum)

Example 4

Prove the formula of 1-norm.

(Proof) According to the definition of vector's 1-norm,

$$\|Au\|_1 = \sum_{i=1}^{m} \left| \sum_{j=1}^{n} a_{ij} u_j \right| \le \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}||u_j| = \sum_{j=1}^{n} (\sum_{i=1}^{m} |a_{ij}|)|u_j|$$

$$\le \max_{1 \le j \le n} \sum_{i=1}^{m} |a_{ij}| \sum_{j=1}^{n} |u_j| = \max_{1 \le j \le n} \sum_{i=1}^{m} |a_{ij}| \, \|u\|_1$$

$$\Rightarrow \frac{\|Au\|_1}{\|u\|_1} \le \max_{1 \le j \le n} \sum_{i=1}^{m} |a_{ij}|$$

This inequality is true for arbitrary vector u. So when the left side takes the supremum w.r.t to u, the inequality is still satisfied. That is, $\|A\|_1 \le \max_j \sum_{i=1}^{m} |a_{ij}|$.

Next, assume that the column sum takes the maximum at the j^*-th column, i.e.

$$\sum_{i=1}^{m} |a_{ij^*}| = \max_{1 \le j \le n} \sum_{i=1}^{m} |a_{ij}|.$$

Set $u_* = e_{j^*}$, then $\|u_*\| = 1$ and

$$\|Au_*\|_1 = \sum_{i=1}^{m} |a_{ij^*}| = \max_{1 \le j \le n} \sum_{i=1}^{m} |a_{ij}| = \max_{1 \le j \le n} \sum_{i=1}^{m} |a_{ij}| \, \|u_*\|_1$$

$$\Rightarrow \frac{\|Au_*\|_1}{\|u_*\|_1} = \max_{1 \le j \le n} \sum_{i=1}^{m} |a_{ij}| \;\Rightarrow \|A\|_1 \ge \frac{\|Au_*\|_1}{\|u_*\|_1} = \max_{1 \le j \le n} \sum_{i=1}^{m} |a_{ij}|.$$

So, we have

$$\|A\|_1 = \max_{1 \le j \le n} \sum_{i=1}^{m} |a_{ij}|.$$

Inner Product of Matrix

- Inner product of $A, B \in \mathbb{F}^{m\times n}$: ($\mathrm{Tr}$ denotes trace)

$$\langle A, B\rangle = \mathrm{Tr}(A^* B). \tag{16}$$

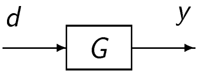
- Why the inner product of matrix is defined as such?
- Denote the ith columns of A, B by a_i, b_i. Then,

$$\mathrm{Tr}(A^* B) = \sum_{i=1}^{n} a_i^* b_i = \mathrm{vec}(A)^*\mathrm{vec}(B) = \langle \mathrm{vec}(A), \mathrm{vec}(B)\rangle . \tag{17}$$

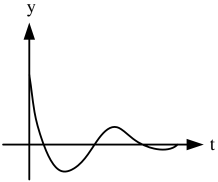
- Inner product of matrices is equal to inner product of the vectors formed by their resp columns.

Signal Norm

- How to measure a response?
- Good candidates
  - Absolute area
  - Maximal magnitude
  - Squared area



(a) System

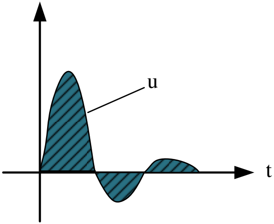


(b) Disturbance response

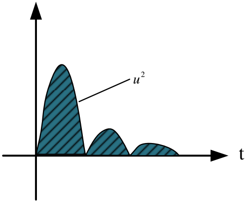
Figure: Disturbance attenuation

Frequently used signal norm

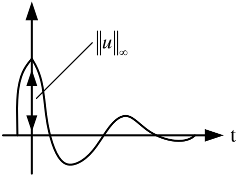
1. 1-norm (Figure (a).): $\|u\|_1 = \int_0^\infty |u(t)|dt$
2. 2-norm (Figure (b).): $\|u\|_2 = \sqrt{\int_0^\infty u^2(t)dt}$
3. Infinity-norm (Figure (c).): $\|u\|_\infty = \sup_{t\in[0,\infty)} |u(t)|$



(a) 1-norm of signal



(b) 2-norm of signal



(c) Infinity-norm of signal

Figure: Signal norms

An example

Example 5

Calculate the norms for the signal

$$u(t) = e^{-3t}, \quad t \geq 0.$$

(Solution) Calculation following the definitions yields

$$\|u\|_1 = \int_0^\infty e^{-3t} dt = \frac{1}{3}, \quad \|u\|_2 = \sqrt{\int_0^\infty e^{-6t} dt} = \frac{\sqrt{6}}{6},$$
$$\|u\|_\infty = \max_{t \geq 0} |e^{-3t}| = 1.$$

Clearly, the values of various norms are different. ∇

Inner Product of Signals

1. Inner product for quadratically integrable signals $u(t), v(t)$

$$\langle u, v \rangle = \int_0^{\infty} u(t)v(t)dt. \tag{18}$$

2. Inner product and norm of periodic functions

$$\langle u, v \rangle = \frac{2}{T}\int_0^{T} u^T(t)v(t)dt, \tag{19}$$

$$\|u\| = \sqrt{\langle u, u \rangle} = \sqrt{\frac{2}{T}\int_0^{T} u^T(t)u(t)dt}. \tag{20}$$

3. How to describe the phase difference of sine waves?

Phase difference of Signals

1. $\sin(\omega t + \varphi)$: projection on the vertical axis of a vector rotating counterclockwise with an angular velocity ω from an initial angle φ.
2. Phase difference of $\sin(\omega t)$ and $\sin(\omega t - \varphi)$ can be thought of as the angle between two vectors rotating at the same speed.
3. How to use the inner product to express the phase angle?

Example 6

Look at $u(t) = A\sin(\omega t), v(t) = B\sin(\omega t - \varphi)$.

$$\langle u, v\rangle = AB\frac{2}{T}\int_0^T \sin(\omega t)\sin(\omega t - \varphi)dt = AB\cos\varphi$$

$$\|u\| = A, \quad \|v\| = B \quad \Rightarrow \cos\varphi = \frac{\langle u, v\rangle}{\|u\|\,\|v\|}.$$

Phase difference between two sine waves indeed has the same meaning as the angle between two vectors in vector space.

Norm and Inner Product of Signals in Frequency Domain

- Inner product

$$\left\langle \hat{f}, \hat{g} \right\rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}^*(j\omega)\hat{g}(j\omega)d\omega. \tag{21}$$

- Norm

$$\|\hat{u}\|_2 = \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}^*(j\omega)\hat{u}(j\omega)d\omega.} \tag{22}$$

Norm and Inner Product of Signals in Frequency Domain

Lemma 2

Assume that vector functions $\hat{f}(s)$, $\hat{g}(s)$ are quadratically integrable. Then, the following statements hold.

1. $\left\langle a\hat{f}, b\hat{g} \right\rangle = \bar{a}b \left\langle \hat{f}, \hat{g} \right\rangle$ *for arbitrary* $a, b \in \mathbb{C}$;
2. $\left\langle \hat{f}, \hat{f} \right\rangle = \left\| \hat{f} \right\|_2^2$;
3. $\left\langle \hat{f}, H\hat{g} \right\rangle = \left\langle H^{\sim}\hat{f}, \hat{g} \right\rangle$ *when* $H(s)$ *has no poles on* $j\omega$ *axis;*
4. *If* $A^*(j\omega)A(j\omega) = I \; (\forall\omega)$, *then* $\left\| A\hat{f} \right\|_2 = \left\| \hat{f} \right\|_2$.

- (4) means that 2-norm is invariant w.r.t. all-pass function
- Time domain and frequency domain norms/inner products are equal (Parseval's theorem).

Computation of 2-norm and Inner Product

- Inner product
  $\hat{f}(s), \hat{g}(s)$ are both strictly proper, rational functions (vectors) with real coefficients and have no purely imaginary poles

$$\left\langle \hat{f}, \hat{g} \right\rangle = \sum_i \operatorname*{Res}_{\Re(s_i)<0} \hat{f}^T(-s)\hat{g}(s) \tag{23}$$

  Res_{s_i} denotes the residue at the point s_i.

- Norm

$$\|g\|_2 = \|\hat{g}\|_2 = \sqrt{\sum_i \operatorname*{Res}_{s_i} \hat{g}^T(-s)\hat{g}(s)}. \tag{24}$$

  s_i denotes a pole of $\hat{g}(s)$.

Computation of 2-norm and Inner Product

Example 7

Let $u(t)$ be the input of $G(s) = 1/(s+1)$, $y(t)$ be the output.

(1) *Calculate the 2-norm of unit impulse response $g(t)$;*

(2) *For $u(t) = e^{-5t}$, compute $\|y\|_2$.*

(Solution) (1) First, $G(-s)G(s) = 1/(1-s)(1+s)$ has a pole $p = -1$ on the left half-plane. The residue at the pole is

$$\lim_{s\to -1}(s+1)G(-s)G(s) = \lim_{s\to -1}(s+1)\frac{1}{(1-s)(s+1)} = \frac{1}{2} \Rightarrow \|g\|_2 = 1/\sqrt{2}.$$

(2) Since $\hat{u}(s) = 1/(s+5)$, $\hat{y}(s)$ is equal to $1/(s+1)(s+5)$ and has two poles $p = -1, -5$. The residues at the stable poles are

$$\lim_{s\to -1}(s+1)\hat{y}(-s)\hat{y}(s) = \frac{1}{48}, \quad \lim_{s\to -5}(s+5)\hat{y}(-s)\hat{y}(s) = -\frac{1}{240}.$$

So, $\|y\|_2 = 1/\sqrt{60}$.

Orthogonality of functions

Lemma 3

Assume that $\hat{f}(s)$ is a stable (vector) function, $\hat{g}(s)$ is an antistable (vector) function without purely imaginary poles, both being strictly proper. Then, there hold

$$\left\langle \hat{f}, \hat{g} \right\rangle = 0, \quad \left\| \hat{f} + \hat{g} \right\|_2^2 = \left\| \hat{f} \right\|_2^2 + \|\hat{g}\|_2^2 .$$

This lemma shows that stable function is orthogonal to antistable function.

(Proof) As $\hat{f}, \hat{g}$ are strictly proper and have no imaginary poles, their inner product exists. Also, since $\hat{f}^T(-s)\hat{g}(s)$ has no poles on the left-half plane, according to (23) $\left\langle \hat{f}, \hat{g} \right\rangle = 0$ is true. The second equation comes from Pythagoras theorem (refer to Theorem 1(3)). •

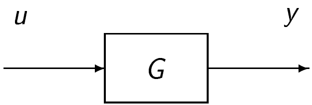
System Norm

- $\mathcal{H}_2$ norm:

$$\|G\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} |G(j\omega)|^2 d\omega} = \sqrt{\int_0^{\infty} g^2(t)dt}. \tag{25}$$

$g(t) = \mathcal{L}^{-1}[G(s)]$: unit impulse response of transfer function $G(s)$

- Implication: $\mathcal{H}_2$ norm is the square root of the squared area of the frequency response gain, equal to the square root of the squared area of unit impulse response.

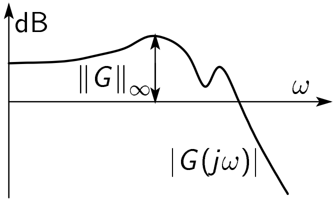


System Norm

- $\mathcal{H}_\infty$ norm:

$$\|G\|_\infty = \sup_{\omega\in(-\infty,\infty)} |G(j\omega)| \tag{26}$$

- $\mathcal{H}_\infty$ norm is the maximal amplitude of the frequency response of transfer function.



$\mathcal{H}_\infty$ norm

Example

Example 8

Calculate the $\mathcal{H}_2$ norm and $\mathcal{H}_\infty$ norm of stable transfer function

$$G(s) = \frac{10}{(s+1)(s+10)}.$$

(Solution) First, we calculate the unit impulse response of $G(s)$.

$$G(s) = \frac{10}{9}\left(\frac{1}{s+1} - \frac{1}{s+10}\right) \Rightarrow g(t) = \frac{10}{9}\left(e^{-t} - e^{-10t}\right), \quad t \geq 0.$$

Thus, we get

$$\|G\|_2 = \sqrt{\int_0^\infty |g(t)|^2 dt} = \sqrt{\frac{5}{11}}.$$

Example

On the other hand,

$$|G(j\omega)|^2 = \frac{100}{(\omega^2+1)(\omega^2+100)}$$

The solutions of

$$0 = \frac{d|G(j\omega)|^2}{d\omega} = \frac{d|G(j\omega)|^2}{d\omega^2}\frac{d\omega^2}{d\omega} = 2\omega\frac{d|G(j\omega)|^2}{d\omega^2}$$

are $\omega = 0$ and $\omega = \infty$. Since $|G(j\infty)| = 0$, we finally get

$$\|G\|_\infty = |G(j0)| = 1.$$