Chapter 20

Gain-Scheduled Control

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Introduction

1. A nonlinear system can be described as an LPV model.
2. Time-varying coefficients of an LPV model are functions of some states, they can be calculated when these states are measured.
3. Therefore, it is possible to change the controller gain according to the variation of coefficients in the LPV model.
4. Can control the LPV system more effectively than controllers with fixed gains.
5. This method is called the *gain-scheduled control*

General Structure

1. LPV model

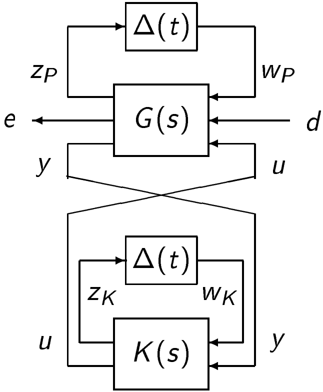
$$\dot{x} = A(p(t))x + B(p(t))u \tag{1}$$
$$y = C(p(t))x. \tag{2}$$

2. Gain-scheduled controller

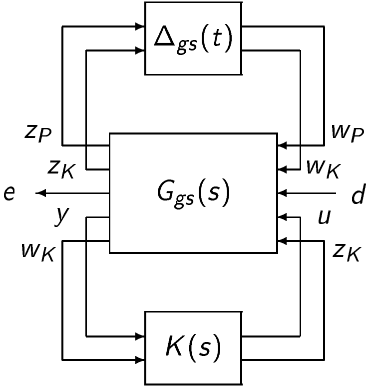
$$\dot{x}_K = A_K(p(t))x_K + B_K(p(t))y \tag{3}$$
$$u = C_K(p(t))x_K + D_K(p(t))y. \tag{4}$$

3. Parameters of controller are changed together with that of the time-varying parameter vector $p(t)$.
4. However, without specifying the structure about the parameter vector $p(t)$, concrete design method cannot be established.

LFT Type Parametric Model



(a) Structure



(b) Equivalent transformation

Figure: LFT type gain-scheduled control system

LFT Type Parametric Model

1. Nominal transfer matrix $G(s)$

$$\dot{x} = Ax + B_1 w_P + B_2 d + B_3 u \tag{5}$$
$$z_P = C_1 x + D_{11} w_P + D_{12} d + D_{13} u \tag{6}$$
$$e = C_2 x + D_{21} w_P + D_{22} d + D_{23} u \tag{7}$$
$$y = C_3 x + D_{31} w_P + D_{32} d \tag{8}$$

2. d: disturbance, e: performance output, y: measured output and u: control input
3. Structure of uncertainty $\Delta(t)$

$$\Delta(t) = \mathrm{diag}(\delta_1(t) I_{r_1},\ \delta_2(t) I_{r_2},\ \cdots,\ \delta_q(t) I_{r_q}), \quad |\delta_i(t)| \le 1. \tag{9}$$

4. Input-output relationship

$$w_P = \Delta(t) z_P. \tag{10}$$

Controller Structure

1. Introducing the same LFT structure about $\Delta(t)$ into the gain-scheduled controller

$$\dot{x}_K = A_K\, x_K + B_{K1}\, w_K + B_{K2}\, y \tag{11}$$

$$z_K = C_{K1}\, x_K + D_{K11}\, w_K + D_{K12}\, y \tag{12}$$

$$u = C_{K2}\, x_K + D_{K21}\, w_K + D_{K22}\, y. \tag{13}$$

2. Gain-scheduling signal w_K

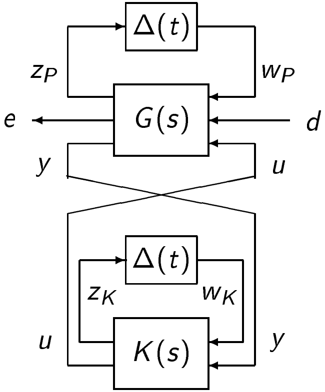
$$w_K = \Delta(t) z_K. \tag{14}$$

3. Signal w_K relies on $\Delta(t)$, it changes the gain of the controller online.
4. Clearer relation

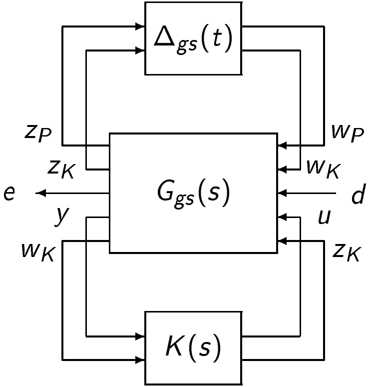
$$\hat{u}(s) = [K_{22} + K_{21}\Delta(I - \Delta K_{11})^{-1} K_{12}]\hat{y}(s). \tag{15}$$

$K_{ij}(s)$: a block of the 2×2 block partition of $K(s)$.

LFT Type Parametric Model



(a) Structure



(b) Equivalent transformation

Figure: LFT type gain-scheduled control system

Equivalent Transformation

1. Figure (b): equivalent system
2. State equation of $G_{gs}(s)$

$$\left[\begin{array}{c} \dot{x} \\ \hline z_K \\ z_P \\ e \\ \hline w_K \\ y \end{array}\right] = \left[\begin{array}{c|cccc|c} A & 0 & B_1 & B_2 & 0 & B_3 \\ \hline 0 & 0 & 0 & 0 & I & 0 \\ C_1 & 0 & D_{11} & D_{12} & 0 & D_{13} \\ C_2 & 0 & D_{21} & D_{22} & 0 & D_{23} \\ \hline 0 & I & 0 & 0 & 0 & 0 \\ C_3 & 0 & D_{31} & D_{32} & 0 & 0 \end{array}\right] \left[\begin{array}{c} x \\ \hline w_K \\ w_P \\ d \\ \hline z_K \\ u \end{array}\right]. \tag{16}$$

3. Dilated uncertainty

$$\Delta_{gs}(t) = \begin{bmatrix} \Delta(t) & \\ & \Delta(t) \end{bmatrix}. \tag{17}$$

4. Design methods: small gain based $\mathcal{H}_\infty$ control, μ synthesis, scaled $\mathcal{H}_\infty$ control and so on.

Gain-scheduled $\mathcal{H}_\infty$ Control Design with Scaling

1. Performance spec

$$\sup_{\|d\|_2 \neq 0} \frac{\|e\|_2}{\|d\|_2} = \|H_{ed}\|_\infty < 1. \tag{18}$$

2. Equivalence to the robust stability of the CLS when a virtual uncertainty $\Delta_P(s)$ ($\|\Delta_P\|_\infty \leq 1$) is inserted between d and e

$$\mathrm{diag}(\Delta_{gs},\ \Delta_P) = \mathrm{diag}(\Delta,\ \Delta,\ \Delta_P).$$

3. Scaled small-gain condition

$$\left\| \begin{bmatrix} L_{gs}^{1/2} & \\ & I \end{bmatrix} \mathcal{F}_\ell(G_{gs}, K) \begin{bmatrix} L_{gs}^{-1/2} & \\ & I \end{bmatrix} \right\|_\infty < 1 \tag{19}$$

4. Structure of scaling matrix

$$L_{gs} = \begin{bmatrix} L_1 & L_2 \\ L_2^T & L_3 \end{bmatrix}, \quad L_i \Delta = \Delta L_i. \tag{20}$$

Solvability Condition

$$N_X = [B_3^T \ D_{13}^T \ D_{23}^T]_\perp, \ N_Y = [C_3 \ D_{31} \ D_{32}]_\perp.$$

$$N_Y^T \left\{ \begin{bmatrix} YA + A^T Y & YB_1 & YB_2 \\ B_1^T Y & -L_3 & 0 \\ B_2^T Y & 0 & -I \end{bmatrix} + \begin{bmatrix} C_1^T & C_2^T \\ D_{11}^T & D_{21}^T \\ D_{12}^T & D_{22}^T \end{bmatrix} \begin{bmatrix} L_3 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix} \right\} N_Y < 0 \tag{21}$$

$$N_X^T \left\{ \begin{bmatrix} AX + XA^T & XC_1^T & XC_2^T \\ C_1 X & -J_3 & 0 \\ C_2 X & 0 & -I \end{bmatrix} + \begin{bmatrix} B_1 & B_2 \\ D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} J_3 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} B_1^T & D_{11}^T & D_{21}^T \\ B_2^T & D_{12}^T & D_{22}^T \end{bmatrix} \right\} N_X < 0 \tag{22}$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0, \quad \begin{bmatrix} L_3 & I \\ I & J_3 \end{bmatrix} \geq 0, \ L_3 > 0, \ J_3 > 0 \tag{23}$$

Computation of Controller

1. Matrix factorization:

$$MM^T = Y - X^{-1}, \quad N^T N = L_3 - J_3^{-1}. \tag{24}$$

2. Lyapunov matrix P and scaling matrix L:

$$P = \begin{bmatrix} Y & M \\ M^T & I \end{bmatrix}, \ L = \begin{bmatrix} I & N \\ N^T & L_3 \end{bmatrix}, \ L_a = \mathrm{diag}(L, \ I_{n_e}), \ J_a = L_a^{-1}. \tag{25}$$

3. Solve the LMI

$$Q + E^T \mathcal{K} F + F^T \mathcal{K}^T E < 0 \tag{26}$$

to get the coefficient matrix $\mathcal{K}$ of the controller.

Computation of Controller

$$\mathcal{K} = \begin{bmatrix} D_{K11} & D_{K12} & C_{K1} \\ D_{K21} & D_{K22} & C_{K2} \\ B_{K1} & B_{K2} & A_K \end{bmatrix} \tag{27}$$

$$\begin{bmatrix} Q & E^T \\ F & \end{bmatrix} = \left[\begin{array}{ccc|c} \overline{A}^T P + P\overline{A} & P\overline{B}_1 & \overline{C}_1^T & P\overline{B}_2 \\ \overline{B}_1^T P & -L_a & \overline{D}_{11}^T & 0 \\ \overline{C}_1 & \overline{D}_{11} & -J_a & \overline{D}_{12} \\ \hline \overline{C}_2 & \overline{D}_{21} & 0 & \end{array}\right] \tag{28}$$

$$\begin{bmatrix} \overline{A} & \overline{B}_1 & \overline{B}_2 \\ \overline{C}_1 & \overline{D}_{11} & \overline{D}_{12} \\ \overline{C}_2 & \overline{D}_{21} & \end{bmatrix} = \left[\begin{array}{cc|ccc|ccc} A & 0 & 0 & B_1 & B_2 & 0 & B_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \\ \hline 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ C_1 & 0 & 0 & D_{11} & D_{12} & 0 & D_{13} & 0 \\ C_2 & 0 & 0 & D_{21} & D_{22} & 0 & D_{23} & 0 \\ \hline 0 & 0 & I & 0 & 0 & & & \\ C_3 & 0 & 0 & D_{31} & D_{32} & & & \\ 0 & I & 0 & 0 & 0 & & & \end{array}\right]. \tag{29}$$

Case study: Stabilization of a unicycle robot

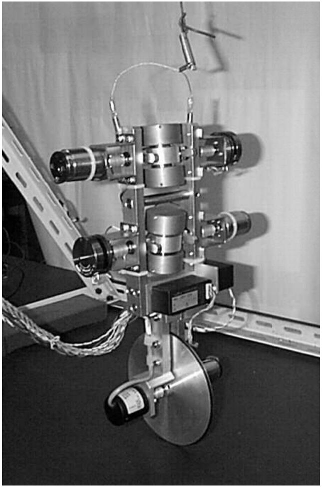


Figure: Unicycle robot in motion

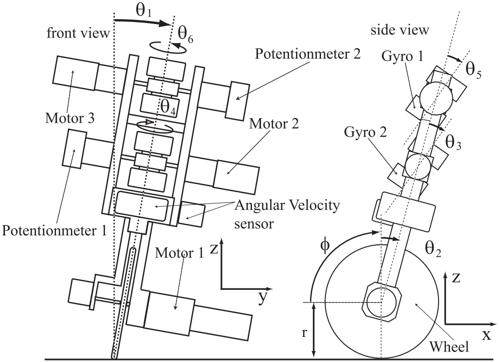


Figure: Front and Side Views of Unicycle Robot

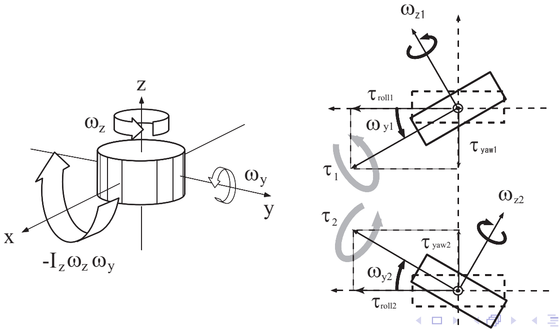
Gyro Actuator

1. Mechanism of torque generation: when a fly-wheel rotating along z axis at speed ω_z is rotated along y axis at speed ω_y, a torque

$$\tau = -I_z \omega_z \omega_y \tag{30}$$

is generated along x axis (left side).

2. τ: called gyro-moment, contributes to lateral stabilization.

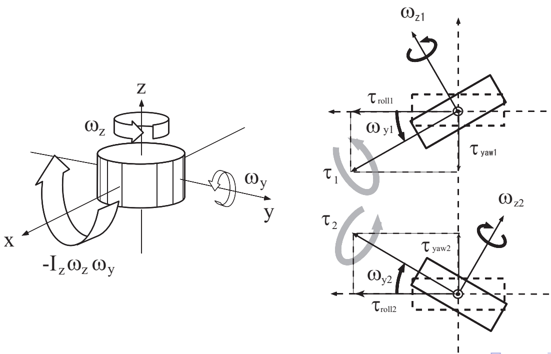


Gyro Actuator

1. When the pitch angle along x axis is θ, the torque τ_{roll} in the lateral direction about x axis is

$$\tau_{\mathrm{roll}} = -I_z \omega_z \cos\theta \omega_y = -R(\theta)\omega_y \tag{31}$$

2. $R(\theta) = I_z \omega_z \cos\theta$: coefficient of gyro-moment.



LPV Model

1. Coefficients of gyro-moments change significantly during the motion, particularly when a lateral force disturbance is applied at the robot

$$R_1(\theta_2,\theta_3) = R_1\cos(\theta_2+\theta_3),\; R_2(\theta_2,\theta_5) = R_2\cos(\theta_2+\theta_5), \tag{32}$$

2. Since $\theta_2 \approx 0$, we have

$$R_1(\theta_2,\theta_3) \approx R_1(\theta_3) = R_1 + \Delta_{R_1}\delta_1(t)$$
$$R_2(\theta_2,\theta_5) \approx R_2(\theta_5) = R_2 + \Delta_{R_2}\delta_2(t)$$

3. LPV model

$$E\dot{x} = \left(A + \sum_{i=1}^{2}\delta_i(t)A_i\right)x + Bu \tag{33}$$
$$y = Cx. \tag{34}$$

Disturbance and Performance Output

1. Disturbance: external force, force disturbance due to power cable, unbalance due to assembly error.
2. Modeled as 2 forces acting at $\ddot{\phi}$ (longitudinal) and $\ddot{\theta}_1$ (lateral)

$$E\dot{x} = \left(A + \sum_{i=1}^{2} \delta_i(t) A_i \right) x + Hd + Bu \tag{35}$$

3. Performance output

$$\bar{e} = [\phi, \theta_3, \theta_5, \dot{\phi}, \dot{\theta}_1, \dot{\theta}_2]^T = Mx$$

4. Tracking error $\phi - r$ for the tracking of reference $r(t)$ by $\phi(t)$;
$\dot{\theta}_1, \dot{\theta}_2$ to keep the posture balance;
θ_3, θ_5 to keep them zeros in the the steady-state.

Generalized Plant

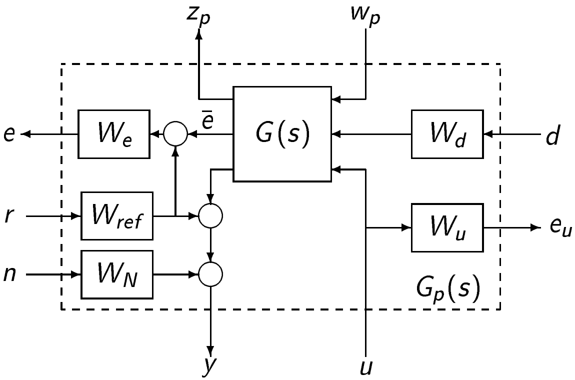


Figure: Block diagram of generalized plant

Generalized Plant and Weights

$$G(s) = \left[\begin{array}{c|ccc} E^{-1}A & E^{-1}L & E^{-1}H & E^{-1}B \\ \hline W & 0 & 0 & 0 \\ M & 0 & 0 & 0 \\ C & 0 & 0 & D \end{array}\right] \tag{36}$$

$$W_e = \mathrm{diag}\left(W_{\phi},\ W_{\theta_3},\ W_{\theta_5},\ W_{\dot{\theta}_1},\ W_{\dot{\theta}_2}\right) \tag{37}$$

$$W_u = \mathrm{diag}\left(W_{\tau_1},\ W_{\tau_2},\ W_{\tau_3}\right) \tag{38}$$

$$W_d = \mathrm{diag}\left(W_{d_1},\ W_{d_2}\right) \tag{39}$$

$$W_{ref} = \begin{bmatrix} -W_r & 0 & 0 & 0 & 0 \end{bmatrix}^T \tag{40}$$

$$W_N = \begin{bmatrix} 0 & 0 & 0 & W_{n_4} & 0 \end{bmatrix}^T \tag{41}$$

1. Spec: for all $\Delta(t)$, ensure

$$\sup_{d_p} \frac{\|e_p\|_2}{\|d_p\|_2} < \gamma \tag{42}$$

2. Constant-scaled small gain condition

$$\|S^{-1}\mathcal{F}_l(G_{gs}, K)SJ_\gamma\|_\infty < 1 \tag{43}$$

$J_\gamma \equiv \mathrm{diag}(I,\ I/\gamma)$, S permutable with $\mathrm{diag}(\Delta,\ \Delta,\ \Delta_p)$.

3. Working ranges presumed as $|\theta_3|, |\theta_5| \leq \frac{\pi}{4}$; perturbation ranges:

$$\Delta_{R_i} = |R_i \times (\cos\frac{\pi}{4} - 1)| \approx R_i \times 0.3, \quad i = 1, 2$$

4. Weights

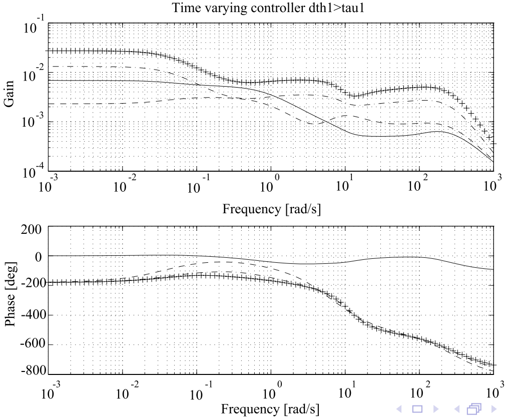
$$W_\phi = \frac{0.2s + 4}{s + 0.0001},\ W_{\theta_3} = W_{\theta_5} = \frac{0.3s + 3}{s + 0.0001},\ W_{\dot{\theta}_1} = W_{\dot{\theta}_2} = 0.01$$

$$W_{\tau_1} = \frac{0.5s + 1}{s + 1000} \times 10^3,\ W_{\tau_2} = W_{\tau_3} = \frac{4s + 40}{s + 2000} \times 10^3$$

$$W_{d_1} = 0.02,\ W_{d_2} = 0.03,\ W_r = 0.1,\ W_n = 0.001$$

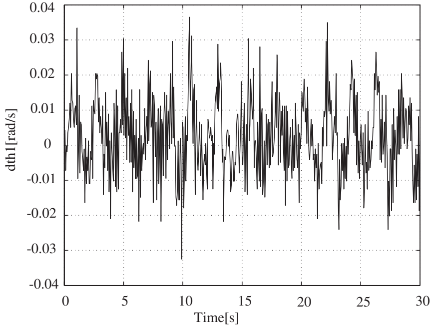
Designed Controller: $\gamma = 0.961$, order=16

1. Significant variation observed in the controllers: $\dot{\theta}_1 \mapsto \tau_1$, $(\phi, \theta_3, \theta_5, \dot{\theta}_2) \mapsto \tau_2$, $(\phi, \theta_3, \theta_5, \dot{\theta}_2) \mapsto \tau_3$.
2. Controller from $\dot{\theta}_1$ to τ_1 shown in the figure

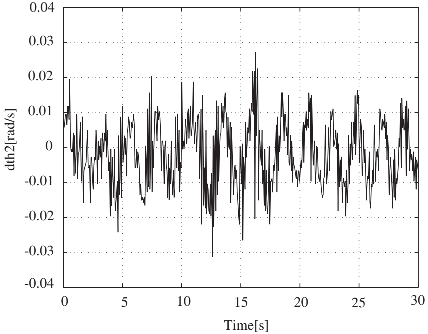


Posture stabilization

1. Zero average velocities, good stability.



(a) Roll speed $\dot{\theta}_1$

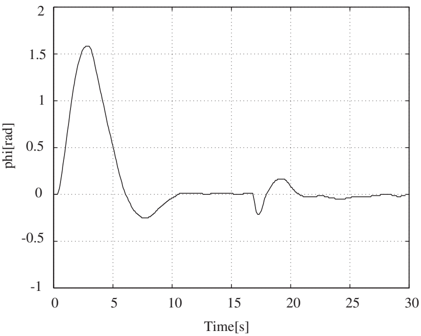


(a) Pitch speed $\dot{\theta}_2$

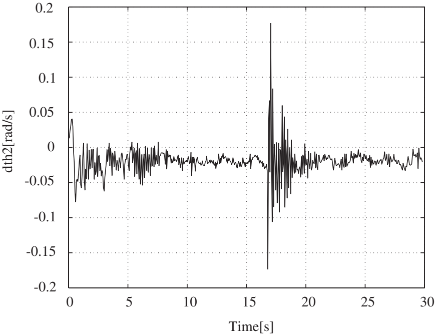
Figure: Posture Stabilization

Disturbance Attenuation

1. Force disturbance applied in the longitudinal and lateral directions at 17 sec and 21 sec.



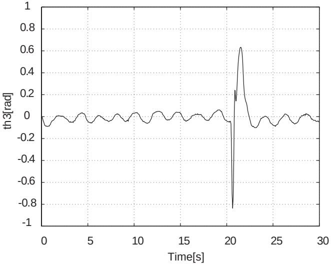
(a) ϕ



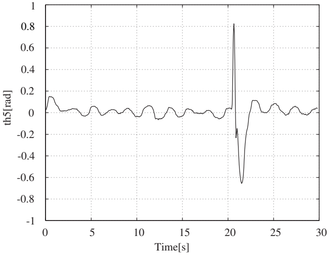
(b) $\dot{\theta}_2$

Figure: Responses to longitudinal force disturbance

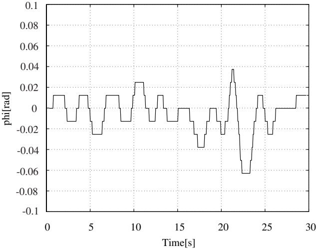
Disturbance Attenuation



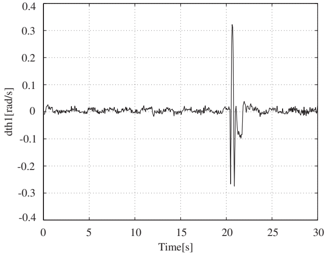
(a) θ_3



(b) θ_5



(c) ϕ



(d) $\dot{\theta}_1$

Running Experiment

1. Move a distance of 40 cm.

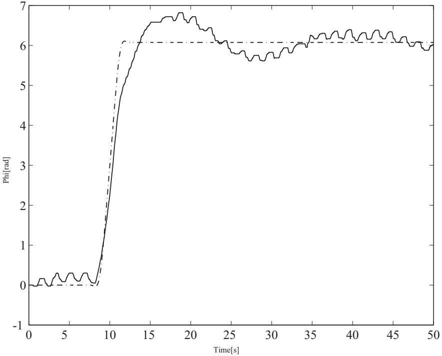


Figure: Longitudinal running: ϕ

Affine LPV Model

$$\dot{x} = A(p(t))x + B_1(p(t))d + B_2(p(t))u \quad (44)$$

$$z = C_1(p(t))x + D_{11}d + D_{12}u \quad (45)$$

$$y = C_2(p(t))x + D_{21}d \quad (46)$$

1. All coefficient matrices are known except the time-varying parameter $p(t)$, such as $A(p) = A_0 + \sum_{i=1}^{q} p_i(t)A_i$.
2. Each time-varying parameter can be measured online, and in a range

$$p_i(\delta) \in [p_{im},\ p_{iM}],\ \ i = 1, \dots, q \quad (47)$$

Gain-Scheduled Controller

$$\begin{aligned} \dot{x}_K &= A_K(p(t))x_K + B_K(p(t))y \\ u &= C_K(p(t))x_K + D_K(p(t))y \end{aligned} \tag{48}$$

$A_K(p) = A_{K0} + \sum_{i=1}^{q} p_i(t)A_{Ki}$, etc.

1. State vector of the closed-loop system: $[x^T \quad x_K^T]^T$
2. State equation of CLS

$$\dot{\xi} = A_c(p)\xi + B_c(p)d, \quad z = C_c(p)\xi + D_c(p)d \tag{49}$$

$$\begin{aligned} A_c(p) &= \begin{bmatrix} A(p) + B_2(p)D_K(p)C_2(p) & B_2(p)C_K(p) \\ B_K(p)C_2(p) & A_K(p) \end{bmatrix} \\ B_c(p) &= \begin{bmatrix} B_1(p) + B_2(p)D_K(p)D_{21} \\ B_K(p)D_{21} \end{bmatrix} \\ C_c(p) &= \begin{bmatrix} C_1(p) + D_{12}D_K(p)C_2(p) & D_{12}C_K(p) \end{bmatrix} \\ D_c(p) &= D_{11} + D_{12}D_K(p)D_{21}. \end{aligned} \tag{50}$$

Easy-to-Design Structure of Gain-Scheduled Controller

Necessary to ensure the coefficient matrices of CLS are affine functions s.t. we can reduce the design to vertex conditions.

- When matrices $(B_2(p), C_2(p))$ both depend on $p(t)$, (B_K, C_K) must be constant matrices besides $D_K = 0$.
- When (B_2, C_2) are both constant matrices, all coefficient matrices of the controller can be affine functions of the scheduling parameter.
- When only B_2 is a constant matrix, (B_K, D_K) must be constant matrices.
- When only C_2 is a constant matrix, (C_K, D_K) must be constant matrices.

Pros and Cons

Merits of affine model

1. A good compatibility with practical systems, greatly simplifies the numerical design.
2. Lyapunov method can be applied and the conservatism is weaker than the small gain method for parametric uncertainty.
3. By the use of common Lyapunov function, it is easy to carry out *multi-objective control* design

Shortcoming: all the time-varying parameters must be known, otherwise the design would be rather difficult.

Robust Multi-Objective Control

1. New variables in variable change method

$$\mathbb{A} = NA_K M^T + NB_K C_2 X + YB_2 C_K M^T + Y(A + B_2 D_K C_2)X$$

$$\mathbb{B} = NB_K + YB_2 D_K, \ \mathbb{C} = C_K M^T + D_K C_2 X, \ \mathbb{D} = D_K.$$

2. When the coefficient matrices of CLS are all affine in $p(t)$,

$$\mathbb{A}(p) = \mathbb{A}_0 + \sum_{i=1}^{q} p_i(t)\mathbb{A}_i, \quad \mathbb{B}(p) = \mathbb{B}_0 + \sum_{i=1}^{q} p_i(t)\mathbb{B}_i \cdots \tag{51}$$

3. Design based on the vertex conditions yields the constant matrices $(\mathbb{A}_i, \mathbb{B}_i, \mathbb{C}_i, \mathbb{D}_i)$ $(i = 0, 1, \ldots, q)$
4. From them we can compute the controller. For example, when (B_2, C_2) are both constant matrices, we have

$$D_{Ki} = \mathbb{D}_i, \ C_{Ki} = (\mathbb{C}_i - D_{Ki} C_2 X)(M^\dagger)^T, \ B_{Ki} = N^\dagger(\mathbb{B}_i - YB_2 D_{Ki})$$

$$A_{Ki} = N^\dagger(\mathbb{A}_i - NB_{Ki} C_2 X - YB_2 C_{Ki} M^T - Y(A_i + B_2 D_{Ki} C_2)X)(M^\dagger)^T. \tag{52}$$

$\mathcal{H}_\infty$ Norm Specification

$||H_{zw}||_\infty < 1$ if

$$\mathrm{He}\begin{bmatrix} A(\theta_j)X + (B_2\mathbb{C})(\theta_j) & A(\theta_j) + (B_2\mathbb{D}C_2)(\theta_j) & B_1(\theta_j) + (B_2\mathbb{D})(\theta_j)D_{21} & 0 \\ \mathbb{A}(\theta_j) & YA(\theta_j) + (\mathbb{B}C_2)(\theta_j) & YB_1(\theta_j) + \mathbb{B}(\theta_j)D_{21} & 0 \\ 0 & 0 & -\frac{1}{2}I & 0 \\ C_1(\theta_j)X + D_{12}\mathbb{C}(\theta_j) & C_1(\theta_j) + D_{12}(\mathbb{D}C_2)(\theta_j) & D_{11} + D_{12}\mathbb{D}D_{21} & -\frac{1}{2}I \end{bmatrix} < 0 \tag{53}$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0. \tag{54}$$

hold for $j = 1, \ldots, N$.

$\mathcal{H}_2$ Norm specification

$\|H_{zw}\|_1 < \gamma$ if

$$\mathrm{He}\begin{bmatrix} A(\theta_j)X + (B_2\mathbb{C})(\theta_j) & A(\theta_j) & 0 \\ \mathbb{A}(\theta_j) & YA(\theta_j) + (\mathbb{B}C_2)(\theta_j) & 0 \\ C_1(\theta_j)X + D_{12}\mathbb{C}(\theta_j) & C_1(\theta_j) & -\frac{1}{2}I \end{bmatrix} < 0 \tag{55}$$

$$\begin{bmatrix} W & B_1(\theta_j)^T & B_1(\theta_j)^T Y \\ B_1(\theta_j) & X & I \\ YB_1(\theta_j) & I & Y \end{bmatrix} > 0 \tag{56}$$

$$\mathrm{Tr}(W) < \gamma^2. \tag{57}$$

hold for $j = 1, \ldots, N$.

Regional Pole Placement

Poles of LPV system are placed to LMI region

$$D = \{z \in \mathbb{C} : L + zM + \bar{z}M^T < 0\}$$

if

$$\begin{aligned}&\mathrm{He}\left(M \otimes \begin{bmatrix} A(\theta_j)X + (B_2\mathbb{C})(\theta_j) & A(\theta_j) + (B_2\mathbb{D}C_2)(\theta_j) \\ \mathbb{A}(\theta_j) & YA(\theta_j) + (\mathbb{B}C_2)(\theta_j) \end{bmatrix} \right) \\ &+ L \otimes \begin{bmatrix} X & I \\ I & Y \end{bmatrix} < 0 \qquad (58)\\ &\begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0. \qquad (59)\end{aligned}$$

hold for $j = 1, \ldots, N$.

Details

Disk centered at $(-c,\ 0)$ and with a radius r

$$\mathrm{He}\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} A(\theta_j)X + (B_2\mathbb{C})(\theta_j) & A(\theta_j) + (B_2\mathbb{D}C_2)(\theta_j) \\ \mathbb{A}(\theta_j) & YA(\theta_j) + (\mathbb{B}C_2)(\theta_j) \end{bmatrix}\right) + \begin{bmatrix} -r & c \\ c & -r \end{bmatrix} \otimes \begin{bmatrix} X & I \\ I & Y \end{bmatrix} < 0 \tag{60}$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0. \tag{61}$$

Details

Half-Plane $\Re(z) < -\sigma$

$$2\sigma \begin{bmatrix} X & I \\ I & Y \end{bmatrix} + \mathrm{He} \begin{bmatrix} A(\theta_j)X + (B_2\mathbb{C})(\theta_j) & A(\theta_j) + (B_2\mathbb{D}C_2)(\theta_j) \\ \mathbb{A}(\theta_j) & YA(\theta_j) + (\mathbb{B}C_2)(\theta_j) \end{bmatrix} < 0 \tag{62}$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0. \tag{63}$$

Details

Sector $\left|\arg z - \pi\right| < \theta$

$$\mathrm{He}\left(\begin{bmatrix} \sin\theta & \cos\theta \\ -\cos\theta & x\sin\theta \end{bmatrix} \otimes \begin{bmatrix} A(\theta_j)X + (B_2\mathbb{C})(\theta_j) & A(\theta_j) + (B_2\mathbb{D}C_2)(\theta_j) \\ \mathbb{A}(\theta_j) & YA(\theta_j) + (\mathbb{B}C_2)(\theta_j) \end{bmatrix} \right) < 0 \tag{64}$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0. \tag{65}$$

Case study: Transient stabilization of power system

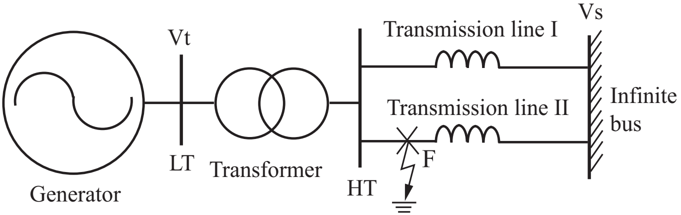


Figure: Single-machine infinite-bus power system

Single-machine infinite-bus power system

$$\dot{\delta} = \omega - \omega_0 \tag{66}$$

$$\dot{\omega} = \frac{\omega_0}{M} P_M - \frac{\omega_0}{M} P_e - \frac{D}{M}(\omega - \omega_0) \tag{67}$$

$$\dot{E}'_q = -\frac{1}{T'_d} E'_q + \frac{x_d - x'_d}{T_{d0} x'_{d\Sigma}} V_s \cos\delta + \frac{1}{T_{d0}} V_f. \tag{68}$$

$$P_e = \frac{E'_q V_s}{x'_{d\Sigma}} \sin\delta - \frac{V_s^2}{2} \frac{x_d - x'_d}{x'_{d\Sigma} x_{d\Sigma}} \sin 2\delta. \tag{69}$$

1. Transient stability: stability in face of short-circuit fault
2. Equilibrium of power system: $(\delta_0,\ \omega_0,\ E'_{q0},\ V_{f0})$.
3. Goal of control: restore the deviated states back to the equilibrium, i.e. the stabilization of error states

$$x_1 = \delta - \delta_0,\ x_2 = \omega - \omega_0,\ x_3 = E'_q - E'_{q0}.$$

LPV model

- Pre-feedback in the field voltage:

$$V_f = \bar{V}_f - \frac{x_d - x_d'}{x_{d\Sigma}'} V_s \cos\delta \tag{70}$$

- Dynamics of internal transient voltage E_q'

$$\dot{E}_q' = \frac{1}{T_{d0}} \left\{ -\frac{x_{d\Sigma}}{x_{d\Sigma}'} E_q' + \frac{x_d - x_d'}{x_{d\Sigma}'} V_s \cos\delta + V_f \right\} = \frac{1}{T_{d0}} \left\{ -\frac{x_{d\Sigma}}{x_{d\Sigma}'} E_q' + \bar{V}_f \right\} \tag{71}$$

- Two time-varying parameters

$$p_1(\delta) = \frac{k_1(\sin\delta - \sin\delta_0) - k_2(\sin 2\delta - \sin 2\delta_0)}{\delta - \delta_0}, \quad p_2(\delta) = \sin\delta$$

LPV model

- 2-parameter LPV model

$$\begin{cases} \dot{x} = A(p)x + B_1 d + B_2 u \\ y = C_2 x. \end{cases} \tag{72}$$

$$A(p) = \begin{bmatrix} 0 & 1 & 0 \\ c_1 p_1(\delta) & c_2 & c_3 p_2(\delta) \\ 0 & 0 & c_5 \end{bmatrix} = A_0 + p_1 A_1 + p_2 A_2. \tag{73}$$

- $u = \bar{V}_f - \bar{V}_{f0}$
- Rotor angle δ is measured.

Multi-Objective Design

1. To add damping to the power system, eigenvalues of the LPV system are placed in a disk region.
2. However, the swing of active power and rotor speed does not fade out quickly enough because of the saturation of field voltage.
3. Rotor angle δ diverges first which causes the divergence of other variables. So, the amplitude of δ should be minimized.

$$\sup_{||d||_2 \neq 0} \frac{||z||_2}{||d||_2} \leq \gamma, \quad z = \delta - \delta_0 = C_1 x. \tag{74}$$

4. Generalized plant

$$G(s) = \left[\begin{array}{c|cc} A(p) & B_1 & B_2 \\ \hline C_1 & 0 & 0 \\ C_2 & 0 & 0 \end{array}\right]. \tag{75}$$

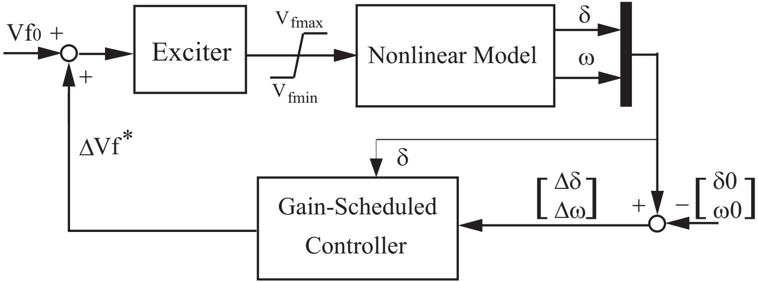
Simulator

1. Dynamics of exciter: a 1st-order transfer function with a limiter:

$$\frac{K_A}{1 + sT_A}, \quad T_A = 0.05, \ K_A = 50$$

2. Limit on the field voltage

$$0.0 \text{ [pu]} \leq V_f(t) \leq 5.0 \text{ [pu]}.$$



Fault sequence and Parameters

- Step1: A fault occurs at $t = 0.0$ sec;
- Step2: Fault is cleared by opening the breakers of the faulted line at $t_F = 0.168$ sec;
- Step3: System operates in a post-fault state.

1. Parameters of the nonlinear power system

$$D = 0.15,\ \ M = 7.00,\ \ T_{d0} = 8.00,\ \ V_s = 0.995$$
$$x_d = 1.81,\ \ x_d' = 0.30,\ \ x_{l1} = 0.5,\ \ x_{l2} = 0.93,\ \ x_T = 0.15.$$

2. Operating point

$$\delta_0 = 0.8807(\approx 50.5^\circ),\ \omega_0 = 314,\ \ E_{q0}' = 1.3228,\ \ V_{f0} = 2.6657.$$

3. Range of δ: assumed to be $[40^\circ, 90^\circ]$ in the design.
4. Disk centered at $(-8, j0)$ and with a radius $r = 6$.

Robustness Test

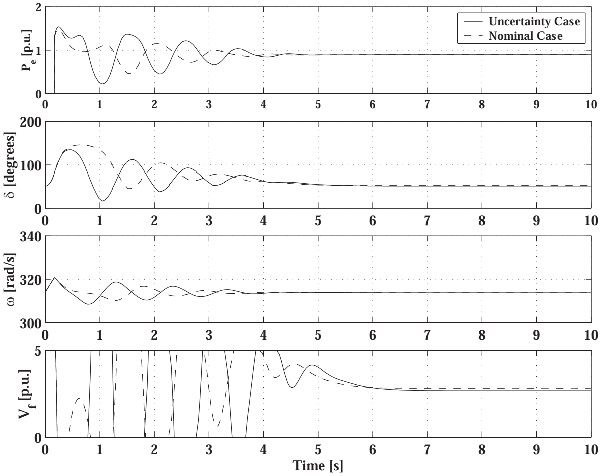


Figure: Robustness test $(\Delta V_s = 0.1V_s,\ \Delta x_L = 0.1x_L,\ \Delta x_T = 0.1x_T)$

Robustness Test

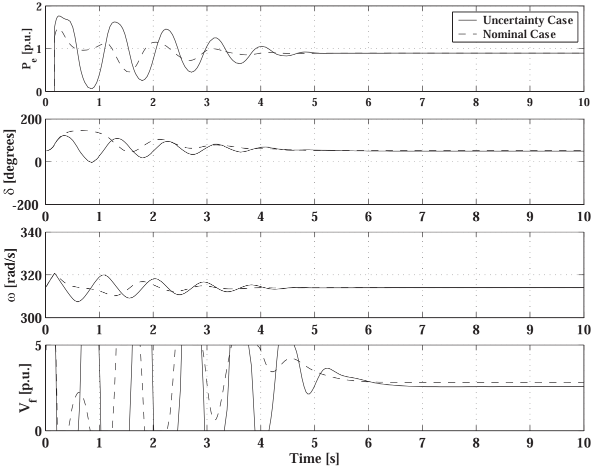


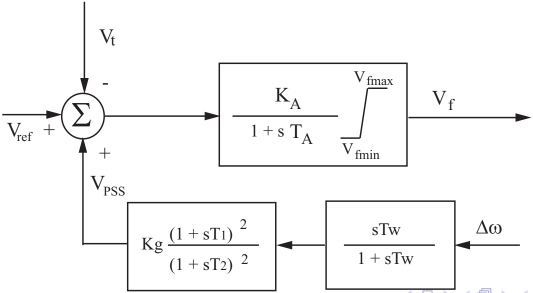
Figure: Robustness test $(\Delta V_s = 0.1V_s,\ \Delta x_L = -0.1x_L,\ \Delta x_T = -0.1x_T)$

PSS

1. Philosophy of PSS: add a damping signal to the AVR (Automatic Voltage Regulator) reference input through a phase lead compensator
2. PSS controller

$$K_{\mathrm{PSS}}(s) = K_g \left(\frac{sT_w}{1 + sT_w} \right) \left(\frac{1 + sT_1}{1 + sT_2} \right)^2 \tag{76}$$

$$K_g = 0.3, \quad T_w = 0.1, \ T_1 = 0.1, \ T_2 = 0.05.$$



Comparison with PSS

1. Gain-scheduled control damps the oscillation faster than the PSS and the oscillation amplitude is smaller.