Chapter 17

μ Synthesis

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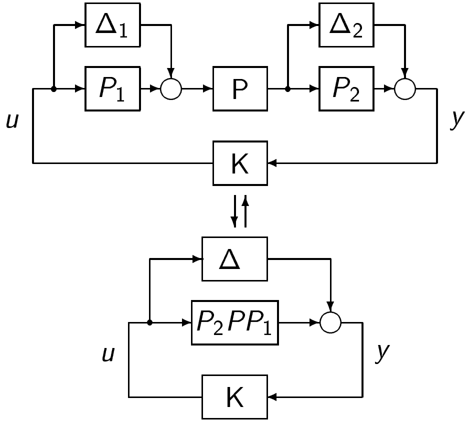
Example 1: When uncertainties put together, we get

$$\hat{y} = (\Delta_2 + P_2)P(\Delta_1 + P_1)\hat{u} = P_2 P P_1 \hat{u} + \Delta\hat{u}$$
$$\Rightarrow \Delta = \Delta_2 P \Delta_1 + \Delta_2 P P_1 + P_2 P \Delta_1. \quad (1)$$



$$\Delta = \Delta_2 P \Delta_1 + \Delta_2 P P_1 + P_2 P \Delta_1. \tag{2}$$

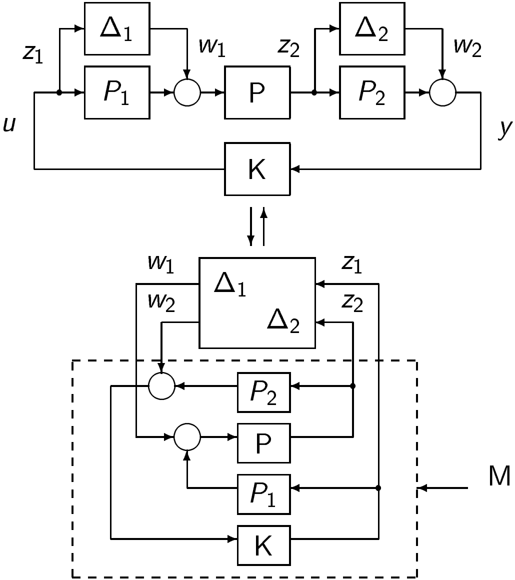
1. Need to estimate an upper bound for Δ based on those of Δ_1, Δ_2.
2. This bound is often enlarged. Further, in the estimated scope of Δ, there are many other uncertainties not belonging to (2). Plant set is far greater than the actual plant set.
3. Control design will often be very conservative. Most typically, the controller gain has to be lowered in the low/middle frequency bands, making it impossible to realize good disturbance attenuation and fast response.

1. Via transformation of block diagram, these two uncertainties can be aggregated as a diagonal matrix $\begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}$ with

$$\begin{bmatrix} \hat{z}_1 \\ \hat{z}_2 \end{bmatrix} = M \begin{bmatrix} \hat{w}_1 \\ \hat{w}_2 \end{bmatrix}, \tag{3}$$

$$M = \begin{bmatrix} KP_2(I - PP_1KP_2)^{-1}P & (I - KP_2PP_1)^{-1}K \\ (I - PP_1KP_2)^{-1}P & PP_1(I - KP_2PP_1)^{-1}K \end{bmatrix}. \tag{4}$$

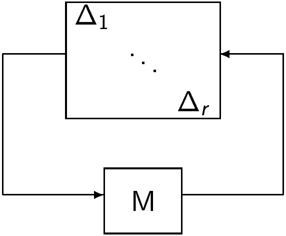
2. This transformation does not change the uncertainties. Therefore, it is possible to achieve a less conservative control design.



Δ_1
Δ_2
z_1
w_1
z_2
w_2
P_1
P
P_2
u
y
K
w_1
z_1
Δ_1
w_2
z_2
Δ_2
P_2
P
P_1
K
M

Structured Uncertainty

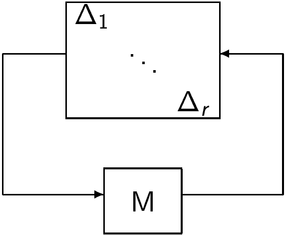
1. In general, when there are r uncertainties Δ_i $(i = 1, \ldots, r)$, the CLS can always be rewritten as



2. Block-diagonal Δ is called *structured uncertainty*.
3. Such transformation does not change the stability of system.

Stability margin

1. Roots of $\det[I - M(s)\Delta(s)] = 0$ are the poles of closed-loop system.
2. $\Delta = 0$: CLS must be stable, $\det[I - M(s)\Delta] = 1 \neq 0$.
3. Next, we fix the dynamics of uncertainty Δ and increase its gain gradually until CLS becomes unstable.
4. Since CLS poles vary continuously with the uncertainty, they must cross the imaginary axis before getting unstable.



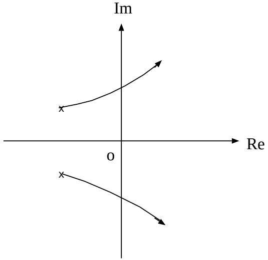
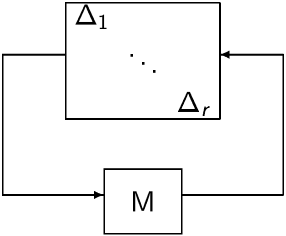


Figure: Continuity of poles

1. Uncertainty destabilizing CLS for the first time must be the one with the smallest norm in all Δ's satisfying

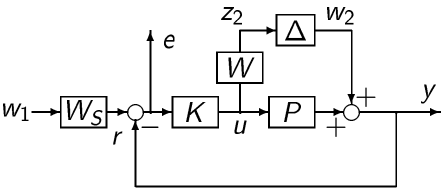
$$\det(I - M(j\omega)\Delta(j\omega)) = 0, \quad \exists \omega \in [0, \infty). \tag{5}$$

2. Its norm is called *stability margin*.
3. Stability margin depends on the diagonal structure of Δ and the matrix M.
4. Reciprocal of stability margin is exactly the structured singular value $\mu_{\mathbf{\Delta}}(M(j\omega))$.

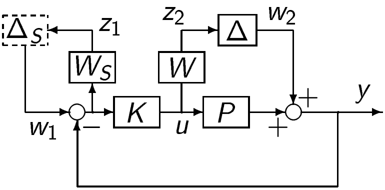


Robust Performance Problem

1. A problem of robust $\mathcal{H}_\infty$ performance can be equivalently converted into a robust stabilization problem of systems with structured uncertainty.
2. This is an even more important motivation for considering the robust stabilization of systems with structured uncertainty.



(a) Original problem



(b) Equivalent stability problem

Definition of μ

1. At a frequency, a transfer matrix becomes a complex matrix.
2. Fix the frequency and consider $\boldsymbol{\Delta} \subset \mathbb{C}^{m \times n}$ and $M \in \mathbb{C}^{n \times m}$.

$$\boldsymbol{\Delta} = \{\Delta \mid \Delta = \mathrm{diag}(\delta_1 I_{r_1}, \ldots, \delta_S I_{r_S}, \Delta_1, \ldots, \Delta_F)\} \qquad (6)$$
$$\delta_i \in \mathbb{C},\ \Delta_j \in \mathbb{C}^{m_j \times n_j}$$

Definition 1

Given matrix $M \in \mathbb{C}^{n \times n}$, the structured singular value $\mu_{\boldsymbol{\Delta}}(M)$ is defined as

$$\mu_{\boldsymbol{\Delta}}(M) = \frac{1}{\min\{\sigma_{\max}(\Delta) \mid \Delta \in \boldsymbol{\Delta},\ \det(I - M\Delta) = 0\}}. \qquad (7)$$

$\mu_{\boldsymbol{\Delta}}(M) = 0$ *when there is no* $\Delta \in \boldsymbol{\Delta}$ *satisfying* $\det(I - M\Delta) = 0$.

Implication of μ

Definition 2

Given $M \in \mathbb{C}^{n\times n}$*, the structured singular value* $\mu_{\mathbf{\Delta}}(M)$ *is defined as*

$$\mu_{\mathbf{\Delta}}(M) = \frac{1}{\min\{\sigma_{\max}(\Delta) \mid \Delta \in \mathbf{\Delta},\ \det(I - M\Delta) = 0\}}. \tag{8}$$

$\mu_{\mathbf{\Delta}}(M) = 0$ *when there is no* $\Delta \in \mathbf{\Delta}$ *satisfying* $\det(I - M\Delta) = 0$.

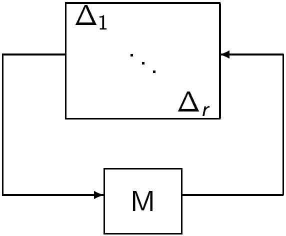
1. $\mu_{\mathbf{\Delta}}(M)$ is the reciprocal of the gain of the smallest uncertainty among all $\Delta \in \mathbf{\Delta}$ satisfying $\det(I - M\Delta) = 0$.
2. Therefore, $\det(I - M\Delta) \neq 0$ holds for all $\Delta \in \mathbf{\Delta}$ satisfying $\sigma_{\max}(\Delta) < 1/\mu_{\mathbf{\Delta}}(M)$.
3. Conversely, as long as there is one $\Delta_1 \in \mathbf{\Delta}$ such that $\sigma_{\max}(\Delta_1) \geq 1/\mu_{\mathbf{\Delta}}(M)$, there must be a $\Delta \in \mathbf{\Delta}$ satisfying $\det(I - M\Delta) = 0$.

Robust Stability Criterion

Theorem 1

Assume that $M(s)$ and the structured uncertainty $\Delta(s) \in \boldsymbol{\Delta}$ are stable, $\|\Delta\|_\infty < \gamma$. Then, the CLS is robustly stable iff

$$\sup_{\omega} \; \mu_{\boldsymbol{\Delta}}(M(j\omega)) \leq \frac{1}{\gamma}. \tag{9}$$



Bounds of $\mu_{\mathbf{\Delta}}(M)$

1. Single scalar block uncertainty $\mathbf{\Delta} = \{\delta I \mid \delta \in \mathbb{C}\}$

$$\mu_{\mathbf{\Delta}}(M) = \rho(M) \tag{10}$$

2. Full block uncertainty $\mathbf{\Delta} = \mathbb{C}^{n \times n}$

$$\mu_{\mathbf{\Delta}}(M) = \sigma_{\max}(M) \tag{11}$$

3. Inclusion relation of uncertainty sets:

$$\{\delta I_n \mid \delta \in \mathbb{C}\} \subset \mathbf{\Delta} \subset \mathbb{C}^{n \times n} \tag{12}$$

4. When an uncertainty is restricted to its subset, a greater uncertainty magnitude is allowed s.t. the corresponding μ gets smaller. Therefore,

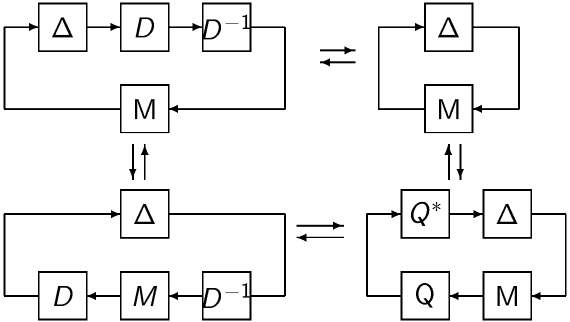
$$\rho(M) \leq \mu_{\mathbf{\Delta}}(M) \leq \sigma_{\max}(M). \tag{13}$$

Improved Bounds of $\mu_{\boldsymbol{\Delta}}(M)$

Scaling matrices

$$\mathbb{D} = \left\{D \mid D = \mathrm{diag}(D_1, \ldots, D_S, d_1 I_{m_1}, \ldots, d_{F-1} I_{m_{F-1}} I_{m_F})\right\} \tag{14}$$

$$\mathbb{Q} = \{Q \in \boldsymbol{\Delta} \mid Q^* Q = I_n\}. \tag{15}$$



Theorem 2

For any $Q \in \mathbb{Q}$ and $D \in \mathbb{D}$, there hold

$$\mu_{\mathbf{\Delta}}(M) = \mu_{\mathbf{\Delta}}(DMD^{-1}) = \mu_{\mathbf{\Delta}}(QM) = \mu_{\mathbf{\Delta}}(MQ). \tag{16}$$

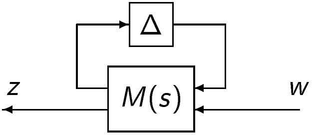
(Proof) First of all, we obtain $\mu_{\mathbf{\Delta}}(M) = \mu_{\mathbf{\Delta}}(DMD^{-1})$ from $\det(I - M\Delta) = \det(I - MD^{-1}D\Delta) = \det(I - MD^{-1}\Delta D) = \det(I - DMD^{-1}\Delta)$.
Secondly, $\mu_{\mathbf{\Delta}}(M) = \mu_{\mathbf{\Delta}}(MQ)$ holds because $\det(I - M\Delta) = 0 \Leftrightarrow \det(I - MQQ^*\Delta) = 0$, $Q^*\Delta \in \mathbf{\Delta}$ and $\sigma_{\max}(Q^*\Delta) = \sigma_{\max}(\Delta)$. Similarly, we can prove that $\mu_{\mathbf{\Delta}}(M) = \mu_{\mathbf{\Delta}}(QM)$. •

1. Improved bounds for $\mu_{\mathbf{\Delta}}(M)$

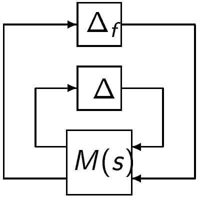
$$\max_{Q \in \mathbb{Q}} \rho(QM) \leq \mu_{\mathbf{\Delta}}(M) \leq \inf_{D \in \mathbb{D}} \sigma_{\max}(DMD^{-1}). \tag{17}$$

2. We may approach $\mu_{\mathbf{\Delta}}(M)$ by solving optimization problems about the spectral radius and the largest singular value.

Robust $\mathcal{H}_\infty$ Performance Condition



(a) Robust performance problem



(b) Equivalent robust stability problem

Theorem 3

Suppose that the uncertainty $\Delta(s) \in \boldsymbol{\Delta}$ *is stable and satisfies* $\|\Delta\|_\infty < 1$. *CLS satisfies* $\|\mathcal{F}_u(M, \Delta)\|_\infty \leq 1$ *for all uncertainties iff*

$$\sup_{\omega \in \mathbb{R}} \mu_{\boldsymbol{\Delta}_P}(M(j\omega)) \leq 1. \tag{18}$$

D-K Iteration Design

1. Improved bounds of μ

$$\max_{Q\in\mathbb{Q}} \rho(QM) \leq \mu_{\mathbf{\Delta}}(M) \leq \inf_{D\in\mathbb{D}} \sigma_{\max}(DMD^{-1}) \tag{19}$$

2. Maximization the lower bound is not convex.
3. Upper bound is the largest singular value and its minimization problem is convex.
4. Convexity of minimization of the largest singular value
   - Minimizing $\sigma_{\max}(DMD^{-1})$ is equivalent to minimizing $\gamma > 0$ satisfying

$$(DMD^{-1})^*(DMD^{-1}) \leq \gamma^2 I \;\Leftrightarrow\; M^*XM \leq \gamma^2 X,\; X = D^*D. \tag{20}$$

   - Minimizing γ subject to this LMI is a GEVP and convex:

$$\begin{gathered}\min \gamma \\ \text{subject to (20)}\end{gathered}$$

   - D is computed by using the singular value decomposition method.

Procedure of D-K Iteration Design

1. CLS transfer matrix M

$$M(s) = \mathcal{F}_\ell(G, K). \tag{21}$$

2. Taking the maximum of $\sigma_{\max}(DMD^{-1})$ w.r.t. all frequencies, the largest singular value $\sigma_{\max}$ becomes the $\mathcal{H}_\infty$ norm.

$$\sup_\omega \mu_{\boldsymbol{\Delta}}(M) \leq \inf_{D \in \mathcal{D}} \left\| DMD^{-1} \right\|_\infty \tag{22}$$

3. Scaling matrix D and controller $K(s)$ need be solved alternately.
4. Idea: when the controller $K(s)$ is known, $M(s)$ is also fixed. So, the scaling matrix $D(s)$ can be calculated pointwise.
5. When the scaling matrix $D(s)$ is given, the controller $K(s)$ can be obtained by solving an $\mathcal{H}_\infty$ control problem.
6. After each iteration the scaling function is added to the generalized plant, which leads to a very high order of the final controller. Model reduction is necessary before the controller is implemented.

Procedure of D-K Iteration Design

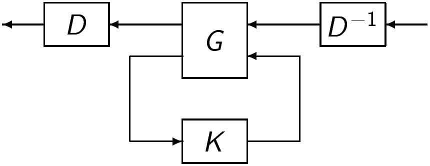
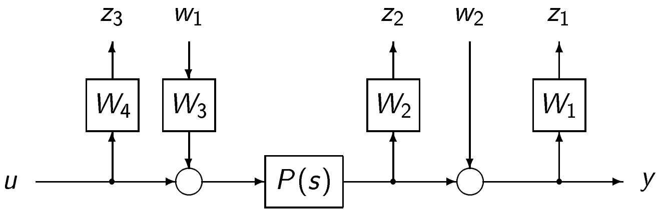


Figure: μ synthesis using scaling

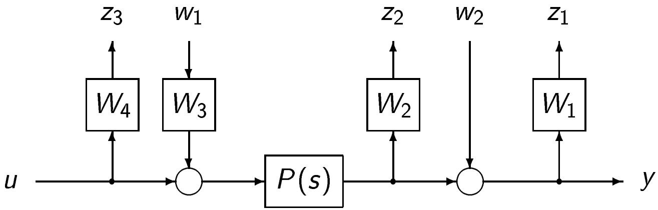
Case Study: $\mathcal{H}_\infty$ control of HDD

1. Head positioning in face of wind disturbance
2. Wind disturbance: step signal
3. w_1 and z_1: input and output used to penalize the disturbance response
4. w_2 and z_2: output and input of multiplicative uncertainty
5. z_3: performance output used to penalize the control input u



Case Study: $\mathcal{H}_\infty$ control of HDD

1. W_1: dynamics of the disturbance
2. W_2: gain of the multiplicative uncertainty
3. W_3: parameter mainly used to tune the response speed
4. W_4: weighting function used to adjust the control input

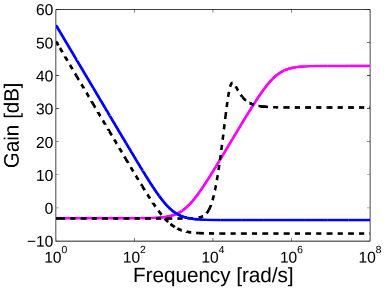


- Gain of performance weight $W_1(s)$ is 60% higher.

$$W_1(s) = 1.6 \times \frac{s + 1.1 \times 8.1 \times 10^2}{s + 1.0 \times 10^{-6}} \times 4.1 \times 10^{-4}$$

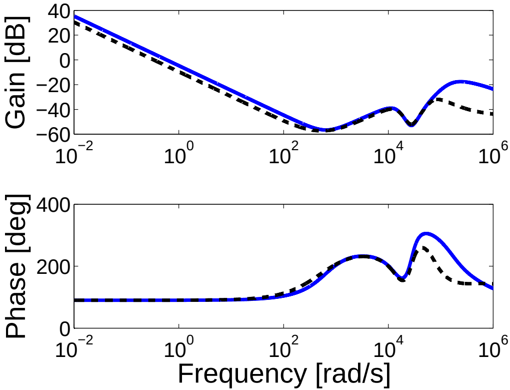
- Transitions of μ and $\mathcal{H}_\infty$ norm

Number of D-K iterations	1	2
μ	1.378	0.998
$\mathcal{H}_\infty$ norm	1.432	0.999



μ controller

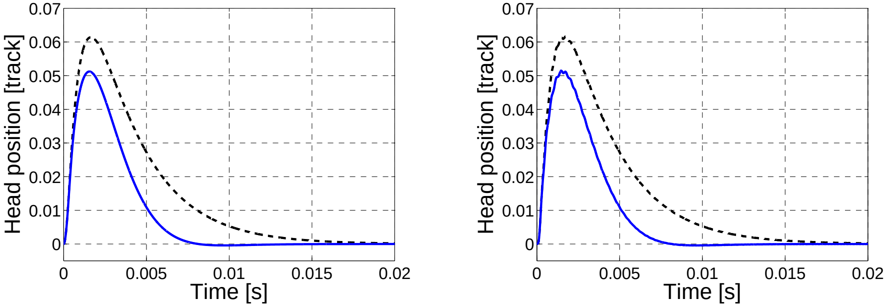
- Higher low frequency gain



Comparison of μ and $\mathcal{H}_\infty$ controllers (solid: μ, dashed: $\mathcal{H}_\infty$)

Output Response

- Smaller amplitude, faster convergence.
- No much difference in the nominal and robust responses.



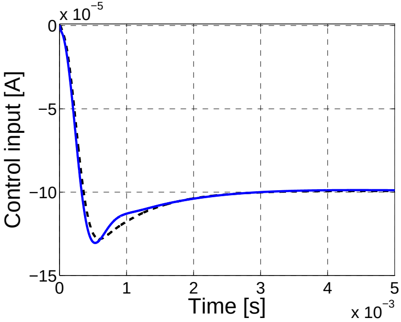
(a) Nominal output response

(b) Actual output response

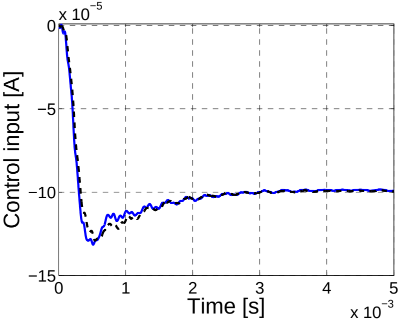
Figure: Step disturbance response (solid: μ, dashed: $\mathcal{H}_\infty$)

Input Response

- Roughly the same amplitude
- Faster amplitude change in 0.5-1.0 sec.



(a) Nominal input response



(b) Actual input response

Figure: Step disturbance response (solid: μ, dashed: $\mathcal{H}_\infty$)