Chapter 8

Relation Between Time Domain and Frequency Domain Properties

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Parseval's Theorem

Theorem 1 (Parseval's theorem)

Signal vectors $f(t), f_1(t), f_2(t) \in \mathbb{R}^n$ *have Fourier transforms* $\hat{f}(j\omega), \hat{f}_1(j\omega), \hat{f}_2(j\omega)$ *resp.*

1. *Inner product in time domain is equal to that in freq domain*

$$\int_0^\infty f_1^T(t) f_2(t) dt = \frac{1}{2\pi} \int_{-\infty}^\infty \hat{f}_1^*(j\omega) \hat{f}_2(j\omega) d\omega. \tag{1}$$

2. *2-norm in time domain is equal to that in freq domain:*

$$\int_0^\infty \|f(t)\|^2 dt = \frac{1}{2\pi} \int_{-\infty}^\infty \left\|\hat{f}(j\omega)\right\|^2 d\omega. \tag{2}$$

$\int_0^\infty \|f(t)\|^2 dt$ represents the energy of signal $f(t)$. In this sense, $\left\|\hat{f}(j\omega)\right\|^2$ can be regarded as the energy density at ω, called power spectrum.

Example

Exponentially convergent signal

$$f(t) = e^{-t} \quad \forall t \geq 0 \;\Leftrightarrow\; \hat{f}(j\omega) = \frac{1}{j\omega + 1}$$

Left side of (2)

$$\int_0^{\infty} (e^{-t})^2 dt = \int_0^{\infty} e^{-2t} dt = \frac{1}{2}.$$

Right side of (2)

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{f}(j\omega)|^2 d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\omega^2 + 1} d\omega = \frac{1}{2\pi} \arctan \omega \Big|_{-\infty}^{\infty} = \frac{1}{2}.$$

Obviously, both sides are equal.

Example

In this example, the power spectrum is $|\hat{f}(j\omega)|^2 = 1/(\omega^2+1)$. It is clear that the energy mainly concentrates in the low frequency band.

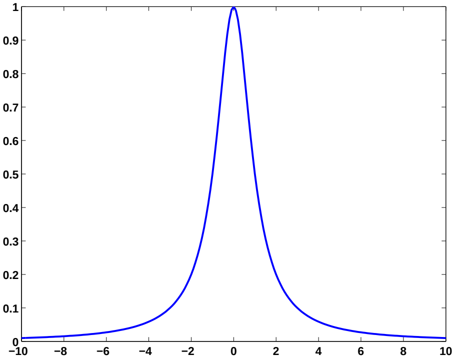


Figure: Power spectrum

System implication

1. In control design, it is very important to fully grasp the power spectrum of a signal.
2. When the signal is a disturbance, the closed-loop system gain needs to be rolled-off in the frequency band where the power spectrum of disturbance is big in order to attenuate its influence on the system output.
3. In this example, this band is roughly $0 \leq \omega \leq 6[\mathrm{rad/s}]$.
4. Finally, a power spectrum is the square of the gain of a signal's frequency response. So we can capture the characteristic of a signal from its gain of frequency response.

KYP Lemma

Theorem 2 (KYP lemma)

Given matrices $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, M = M^T \in \mathbb{R}^{(n+m)\times(n+m)}$. *Assume that* A *has no eigenvalues on the imaginary axis and* (A, B) *is controllable. Then the following statements are equivalent.*

1. *For all* ω *including the infinity, there holds*
$$\begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}^* M \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix} \le 0. \tag{3}$$
2. $\exists P = P^T \in \mathbb{R}^{n\times n}$ *satisfying*
$$M + \begin{bmatrix} A^TP + PA & PB \\ B^TP & 0 \end{bmatrix} \le 0. \tag{4}$$

When both are strict inequalities, the equivalence is still true and (A, B) *needs not be controllable.*

Application in bounded real lemma

1. Gain property about a stable transfer matrix $G(s)$:

$$G^*(j\omega)G(j\omega) < \gamma^2 I \quad \forall\omega \in [0,\ \infty]. \tag{5}$$

2. Equivalent to the $\mathcal{H}_\infty$ norm condition $\|G\|_\infty < \gamma$
3. Equivalent expression

$$G(s) = C(sI - A)^{-1}B + D = [C \quad D]\begin{bmatrix} (sI - A)^{-1}B \\ I \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}^* \begin{bmatrix} C^TC & C^TD \\ D^TC & D^TD \end{bmatrix} \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix} < \gamma^2 I \Leftrightarrow$$

$$\begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}^* \begin{bmatrix} C^TC & C^TD \\ D^TC & D^TD - \gamma^2 I \end{bmatrix} \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix} < 0. \tag{6}$$

Application in bounded real Lemma

- Application of KYP lemma: $\exists P = P^T$ satisfying

$$\begin{bmatrix} C^T C & C^T D \\ D^T C & D^T D - \gamma^2 I \end{bmatrix} + \begin{bmatrix} A^T P + PA & PB \\ B^T P & 0 \end{bmatrix} < 0. \tag{7}$$

Lemma 1 (Bounded real lemma)

Given $G(s) = (A, B, C, D)$, the following statements are equivalent:

1. *A is stable and $\|G\|_\infty < \gamma$;*
2. *There exists a positive definite matrix P satisfying*

$$\begin{bmatrix} A^T P + PA & PB & C^T \\ B^T P & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} < 0. \tag{8}$$

Application in bounded real Lemma

(Proof) Dividing both sides of inequality (7) with γ and renaming P/γ as P, (7) can be written as

$$\begin{bmatrix} A^T P + PA & PB \\ B^T P & -\gamma I \end{bmatrix} - \begin{bmatrix} C^T \\ D^T \end{bmatrix} \cdot (-\gamma I)^{-1} \cdot \begin{bmatrix} C & D \end{bmatrix} < 0.$$

It follows from Schur's lemma that (8) is equivalent to (7). Finally, we need only prove that the stability of A is equivalent to $P > 0$. Since the $(1,1)$ block of (8) is

$$PA + A^T P < 0,$$

the equivalence is immediate by Lyapunov's stability theory. •

Example

$$G(s) = \frac{a}{s^2 + 2s + 2} = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ -2 & -2 & 1 \\ \hline a & 0 & 0 \end{array}\right], \quad a > 0$$

1. Bode plot shows that its largest gain is $a/2$.
2. $\mathcal{H}_\infty$ norm of $G(s)$ is less than 1 (i.e. $\gamma = 1$) iff $a < 2$.
3. When $a = 1.9$, LMI (8) has a positive definite solution

$$P = \begin{bmatrix} 3.8024 & 1.5253 \\ 1.5253 & 1.8265 \end{bmatrix}$$

4. However, when $a \geq 2$, no positive definite solution exists for (8).
5. This implies that we can use the bounded real lemma to calculate the $\mathcal{H}_\infty$ norm of transfer matrices.
6. Calculation of $\mathcal{H}_\infty$ norm: Reduce γ gradually until there is no positive definite solutions for (8). The last γ is the $\mathcal{H}_\infty$ norm.

Time-domain interpretation of bounded real lemma

$$G(s): \ \dot{x} = Ax + Bu, \ y = Cx + Du.$$

Quadratic function

$$V(x) = x^T P x > 0 \tag{9}$$

Multiplying the inequality (7) by $\begin{bmatrix} x \\ u \end{bmatrix}$, we get

$$\begin{aligned}
0 &> x^T(A^T P + PA)x + x^T PBu + u^T B^T Px + x^T C^T Cx \\
&\quad + x^T C^T Du + u^T D^T Cx + u^T(D^T D - I)u \\
&= x^T P(Ax + Bu) + (Ax + Bu)^T Px + (Cx + Du)^T(Cx + Du) - u^T u \\
&= x^T P\dot{x} + \dot{x}^T Px + y^T y - u^T u.
\end{aligned}$$

Since $\dot{V} = x^T P\dot{x} + \dot{x}^T Px$,

$$\dot{V}(x) < u^T u - y^T y \tag{10}$$

Time-domain interpretation of bounded real lemma

After integration, we have

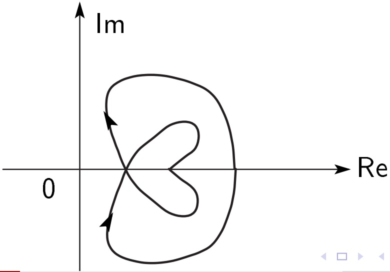
$$V(x(t)) < V(x(0)) + \int_0^t [u^T(\tau)u(\tau) - y^T(\tau)y(\tau)]d\tau. \tag{11}$$

1. $u^T u$, $y^T y$ are the input and output powers, their difference is the power supplied to the system. After integration it becomes the energy supplied to the system.
2. $V(x)$ can be regard as a *storage function* of the system energy.
3. This inequality implies that the variation of the energy stored in a system is less than the energy supplied by the input.
4. A bounded real system consumes a part of the energy supplied by the input. So, it is called a *dissipative system*.

Positive real function

$$G^*(j\omega) + G(j\omega) \geq 0 \quad \forall\omega \in [0,\ \infty] \tag{12}$$

1. Numbers of input and output must be equal
2. Scalar case: left side equals twice of $\Re[G(j\omega)]$, thus it is non-negative.
3. System viewpoint: phase angle of a PR function is limited in $[-90^\circ, 90^\circ]$
4. Relative degree does not exceed 1



Positive real function

- Example: it can be judged from the Nyquist diagram that transfer function $G(s) = 1/(s+1)$ is positive real.
- Some unstable systems may also have a frequency property like (12). For example,

$$G(s) = \frac{s-1}{s-2} \;\Rightarrow\; \Re[G(j\omega)] = \Re\left[\frac{j\omega - 1}{j\omega - 2}\right] = \frac{2+\omega^2}{4+\omega^2} > 0,$$

thus it satisfies the condition (12).

Application in positive real Lemma

Equivalent conditions of inequality (12)

$$\begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}^* \begin{bmatrix} C^T \\ D^T \end{bmatrix} + [C \quad D] \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix} \geq 0.$$

$$-\begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}^* \begin{bmatrix} 0 & C^T \\ C & D + D^T \end{bmatrix} \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix} \leq 0$$

By KYP lemma, when (A, B) is controllable there is a symmetric matrix P satisfying

$$\begin{bmatrix} A^T P + PA & PB \\ B^T P & 0 \end{bmatrix} - \begin{bmatrix} 0 & C^T \\ C & D + D^T \end{bmatrix} \leq 0. \tag{13}$$

Application in positive real Lemma

Lemma 2 (Positive real lemma)

Let (A, B, C, D) be a minimal realization of $G(s)$ and matrix A be stable. Then the following statements are equivalent:

(1) *$G(s)$ satisfies the positive real condition (12);*

(2) *There is a matrix $P > 0$ satisfying inequality (13);*

(3) *There is a matrix $P > 0$ and a full row rank matrix $[L \quad W]$ satisfying*

$$\begin{bmatrix} A^T P + PA & PB \\ B^T P & 0 \end{bmatrix} - \begin{bmatrix} 0 & C^T \\ C & D + D^T \end{bmatrix} = - \begin{bmatrix} L^T \\ W^T \end{bmatrix} [L \quad W]. \quad (14)$$

Example

$$G(s) = \frac{s+a}{s^2+2s+2} = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ -2 & -2 & 1 \\ \hline a & 1 & 0 \end{array}\right].$$

1. Real part of its frequency response

$$\Re[G(j\omega)] = \Re\left[\frac{a+j\omega}{2-\omega^2+j2\omega}\right] = \frac{2a+(2-a)\omega^2}{(2-\omega^2)^2+4\omega^2}.$$

2. For all ω, the condition for $\Re[G(j\omega)] > 0$ is

$$2a > 0, \quad 2-a \geq 0 \Rightarrow 0 < a \leq 2.$$

3. LMI (13) has a positive definite solution

$$P = \begin{bmatrix} 4.0 & 1.0 \\ 1.0 & 1.0 \end{bmatrix}$$

for $a = 1$, but no positive definite solution exists for $a = 3$. This shows that this transfer function is not positive real when $a = 3$.

Strongly positive real function

1. Strict inequality case

$$G^*(j\omega) + G(j\omega) > 0 \quad \forall\omega \in [0, \ \infty] \tag{15}$$

2. $G(s)$ is called a *strongly positive real matrix*
3. Strongly PR requires $G^*(j\infty) + G(j\infty) = D^T + D > 0$, i.e., relative degree of $G(s)$ must be 0.
4. $G(s) = (s + a)/(s^2 + 2s + 2)$ is not strongly PR. But $G(s) = (s + a)/(s + 1)$ is so long as $a > 0$

$$G(j\omega) = \frac{j\omega + a}{j\omega + 1} = \frac{a + \omega^2 + j(1 - a)\omega}{\omega^2 + 1}$$

5. Strongly PR matrix must have full normal rank. That is, for almost all s, $G(s)$ should have full rank.

Strongly positive real lemma

Lemma 3 (Strongly positive real lemma)

For transfer matrix $G(s) = (A, B, C, D)$, the following statements are equivalent:

(1) *Matrix A is stable and $G(s)$ is strongly positive real;*

(2) *There is a positive definite matrix P satisfying the strict inequality*

$$\begin{bmatrix} A^T P + PA & PB \\ B^T P & 0 \end{bmatrix} - \begin{bmatrix} 0 & C^T \\ C & D + D^T \end{bmatrix} < 0. \tag{16}$$

Strictly positive real function

1. Strictly proper $G(s)$: $G(j\infty) = D = 0$ at $s = \infty$, not a strongly positive real function.
2. But, there are many transfer matrixes satisfying $G^*(j\omega) + G(j\omega) > 0$ at all finite frequencies except the infinity, i.e.

$$G^*(j\omega) + G(j\omega) > 0 \quad \forall \omega \in [0,\ \infty). \tag{17}$$

3. Such a stable transfer matrix is called a *strictly positive real matrix*
4. $G(s) = (s + a)/(s^2 + 2s + 2)$ is strictly positive real when $0 < a \leq 2$.

Modified strictly PR function

1. What is the state space condition for a transfer function to be strictly positive real?
2. Unfortunately, this is still an open problem for the strictly positive real matrix defined above.
3. In view of this fact, Narendra-Taylor proposed to use the following frequency domain characteristic to replace (17).

Definition 1 (Modified strictly positive realness)

$G(s)$ is called strictly PR if there exists a constant $\epsilon > 0$ s.t. $G(s-\epsilon)$ is stable and satisfies the PR condition:

$$G^*(j\omega - \epsilon) + G(j\omega - \epsilon) \geq 0 \ \forall \omega \in [0, \ \infty]. \tag{18}$$

Strictly positive real lemma

Lemma 4

Square transfer matrix $G(s) = (A, B, C, D)$ is stable and has full normal rank. Then the following statements are equivalent.

1. *There exists a constant $\epsilon > 0$ such that $G(s-\epsilon)$ is positive real.*
2. *$G^*(j\omega) + G(j\omega) > 0$ holds for any finite frequency ω and*

$$\lim_{\omega \to \infty} \omega^{2\rho} \det[G^*(j\omega) + G(j\omega)] > 0$$

ρ is the dimension of the kernel space of constant matrix $D + D^T$, namely, $\rho = \dim(\mathrm{Ker}(D + D^T))$.

That is, the positive realness of $G(s-\epsilon)$ guarantees that $G(s)$ is strictly positive real.

Example

$$G(s) = \frac{s+a}{s^2+2s+2}.$$

1. For sufficiently small $\epsilon > 0$, $G(s-\epsilon)$ is still stable and

$$\begin{aligned}\Re[G(j\omega-\epsilon)] &= \Re\left[\frac{a-\epsilon+j\omega}{1+(1-\epsilon)^2-\omega^2+j2(1-\epsilon)\omega}\right] \\ &= \frac{(a-\epsilon)[1+(1-\epsilon)^2]+(2-a-\epsilon)\omega^2}{[1+(1-\epsilon)^2-\omega^2]^2+4(1-\epsilon)^2\omega^2}.\end{aligned}$$

2. For all ω including the infinity, $\Re[G(j\omega-\epsilon)] \geq 0$ iff

$$a-\epsilon \geq 0, \quad 2-a-\epsilon \geq 0 \Rightarrow \epsilon \leq a \leq 2-\epsilon.$$

3. There is a small gap between this bound and the strictly PR condition $0 < a \leq 2$. This gap shrinks as $\epsilon \to 0$.

Time-domain interpretation of positive real lemma

$$G(s): \ \dot{x} = Ax + Bu, \ y = Cx + Du$$

Storage function

$$V(x) = x^T P x \tag{19}$$

Multiplying inequality (13) with $[x^T \ u^T]$, we have

$$\begin{aligned}
0 &\le x^T(A^T P + PA)x + x^T PBu + u^T B^T Px - x^T C^T u - u^T Cx \\
&\quad - u^T(D^T + D)u \\
&= x^T P(Ax + Bu) + (Ax + Bu)^T Px - (Cx + Du)^T u - u^T(Cx + Du) \\
&= x^T P\dot{x} + \dot{x}^T Px - y^T u - u^T y \\
&\Rightarrow \dot{V}(x) \le 2y^T u.
\end{aligned}$$

Time-domain interpretation of positive real lemma

Integration leads to

$$V(x(t)) \leq V(x(0)) + 2\int_0^t y^T(\tau)u(\tau)d\tau. \tag{20}$$

1. $y^T u$ is the *supply rate* of the energy injected into the system.
2. Energy stored in the system is less than the energy supplied by the input.
3. Such a system is called a *passive system*.

Time-domain interpretation of positive real lemma

1. Why $y^T u$ is the supply rate of the energy injected into the system?
2. Circuit consisting of an ideal voltage source and a load: u is the voltage of power source, y is the current of load impedance Z. Their product is apparently the power which the power source supplies to the load impedance Z.

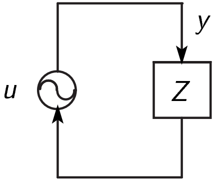


Figure: Energy supply rate for load impedance