Chapter 15

$\mathcal{H}_2$ Control

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$\mathcal{H}_2$ Norm of Transfer Function

1. Transfer matrix
$$G(s) = (A,\ B,\ C,\ D) \tag{1}$$
2. $\mathcal{H}_2$ norm
$$\|G\|_2 := \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{Tr}[G^*(j\omega)G(j\omega)]\, d\omega} \tag{2}$$
3. $g(t)$: inverse Laplace transform of $G(s)$ (namely its impulse response)
4. Alternative form (Parseval's theorem)
$$\|G\|_2 = \|g\|_2 = \sqrt{\int_0^{\infty} \mathrm{Tr}[g^T(t)g(t)]\, dt} \tag{3}$$

Relation with Input and Output

1. SISO: $\mathcal{H}_2$ norm of a transfer function equals the 2-norm of its impulse response.
2. MIMO: consider an orthonormal set of $\{u_i\}$ $(i = 1, \ldots, m)$

$$U^T U = UU^T = I, \quad U = [u_1, \; \cdots, \; u_m].$$

3. Response to impulse input $w_i(t) = u_i \delta(t)$ is $y_i(t) = g(t) u_i$.
4. Then, $\mathcal{H}_2$ norm equals the sum of 2-norm of all responses to orthonormal impulse inputs.

$$\sum_{i=1}^{m} \|y_i\|_2^2 = \sum_{i=1}^{m} \int_0^\infty y_i^T(t) y_i(t)\, dt = \sum_{i=1}^{m} \int_0^\infty u_i^T g^T(t) g(t) u_i \, dt$$

$$= \int_0^\infty \sum_{i=1}^{m} \mathrm{Tr}\left(g^T(t) g(t) u_i u_i^T \right) \, dt = \int_0^\infty \mathrm{Tr}\left(g^T(t) g(t) UU^T \right) \, dt$$

$$= \int_0^\infty \mathrm{Tr}\left(g^T(t) g(t) \right) \, dt = \|G\|_2^2$$

Relation with Input and Output

1. $\mathcal{H}_2$ norm equals the variance of steady-state response to a unit white noise
2. White noise input $u(t)$:
$$\mathrm{E}[u(t)] = 0 \ \forall t, \quad \mathrm{E}[u(t)u^T(\tau)] = \delta(t-\tau)I. \tag{4}$$
3. Variance of y

$$\mathrm{E}\left[y^T(t)y(t)\right] = \mathrm{E}\left[\int_0^t \int_0^t (g(t-\alpha)u^T(\alpha))g(t-\beta)u(\beta)d\alpha d\beta\right]$$
$$= \int_0^t \int_0^t \mathrm{Tr}\left(g^T(t-\alpha)g(t-\beta)\mathrm{E}\left(u(\beta)u^T(\alpha)\right)\right)\, d\alpha d\beta$$
$$= \int_0^t \mathrm{Tr}\left(g^T(t-\beta)g(t-\beta)\right)\, d\beta = \int_0^t \mathrm{Tr}\left(g^T(\tau)g(\tau)\right)\, d\tau$$
$$\Rightarrow \lim_{t\to\infty} \mathrm{E}\left[y^T(t)y(t)\right] = \|G\|_2^2 .$$

Weighting Function vs Disturbance/Noise

1. Disturbance has dynamics $W(s)$. Then $\hat{y}(s) = G(s)W(s)$.
2. Cost function (with weighting function)

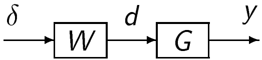
$$\|y\|_2 = \|GW\|_2 \,. \tag{5}$$

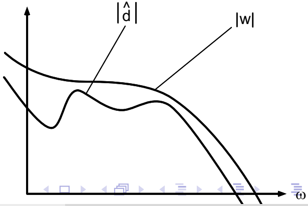
3. Even when only an upper bound $|W(j\omega)|$ is known,

$$|\hat{d}(j\omega)| \leq |W(j\omega)| \quad \forall\ \omega, i.e.$$

disturbance is still suppressed if $\|GW\|_2$ is minimized because

$$\left\|G\hat{d}\right\|_2 \leq \|GW\|_2 \,.$$





Condition for $\|G\|_2 < \gamma$

Lemma 1

The following statements are equivalent.

1. *A is stable and $\left\|C(sI - A)^{-1}B\right\|_2 < \gamma$.*
2. *There is a matrix $X = X^T > 0$ satisfying*

$$XA + A^T X + C^T C < 0, \quad \mathrm{Tr}(B^T XB) < \gamma^2. \tag{6}$$

3. *There exist matrices $X = X^T$ and $W = W^T$ satisfying*

$$\begin{bmatrix} XA + A^T X & C^T \\ C & -I \end{bmatrix} < 0 \tag{7}$$

$$\begin{bmatrix} W & B^T X \\ XB & X \end{bmatrix} > 0, \quad \mathrm{Tr}(W) < \gamma^2. \tag{8}$$

Example: 2-DOF system

1. $\mathcal{H}_2$ norm is ideally suited for rating transient response.
2. Input penalty: in reference tracking, the input is a persistent signal in the steady state. We should limit its rate of change instead.
3. To remove the steady-state value, we use W_r^{-1} to filter the input.
4. For step signal $r(t) = 1(t)$, the input filtered by $W_r^{-1}(s) = s$ becomes its derivative $\dot{u}(t)$.

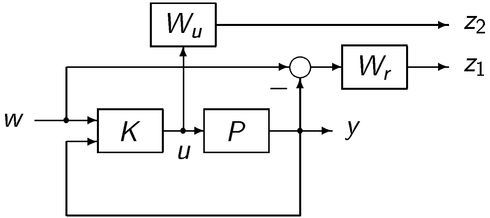


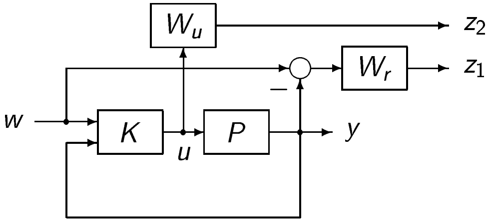
Figure: $\mathcal{H}_2$ tracking control problem of 2-DOF system

Example: 2-DOF system

1. Generalized plant

$$\left[\begin{array}{c} z_1 \\ z_2 \\ \hline w \\ y \end{array}\right] = \left[\begin{array}{c|c} W_r & -W_r P \\ 0 & W_u \\ \hline I & 0 \\ 0 & P \end{array}\right] \left[\begin{array}{c} w \\ u \end{array}\right] = G \left[\begin{array}{c} w \\ u \end{array}\right]$$

2. Minimizing the $\mathcal{H}_2$ norm of CLS from w to $\left[\begin{array}{c} z_1 \\ z_2 \end{array}\right]$, called an $\mathcal{H}_2$ control problem.



$\mathcal{H}_2$ control problem

1. For any given $\gamma > 0$, design a controller satisfying $\|H_{zw}\|_2 < \gamma$.

$$G(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right]. \tag{9}$$

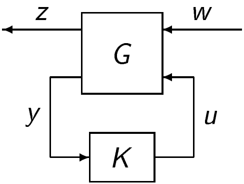


Figure: Generalized feedback system

Solution

Theorem 1

(A, B_2) is stabilizable and (C_2, A) is detectable. Then,

1. *$\mathcal{H}_2$ control is solvable iff $\exists X = X^T$, $Y = Y^T$, $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$, W satisfying*

$$\mathrm{He}\begin{bmatrix} AX + B_2\mathbb{C} & A & 0 \\ \mathbb{A} & YA + \mathbb{B}C_2 & 0 \\ C_1X + D_{12}\mathbb{C} & C_1 & -\frac{1}{2}I \end{bmatrix} < 0 \tag{10}$$

$$\begin{bmatrix} W & B_1^T & B_1^T Y \\ B_1 & X & I \\ YB_1 & I & Y \end{bmatrix} > 0, \quad \mathrm{Tr}(W) < \gamma^2. \tag{11}$$

2. *$\mathcal{H}_2$ controller is given by*

$$A_K = N^{-1}(\mathbb{A} - NB_K C_2 X - YB_2 C_K M^T - YAX)(M^{-1})^T \tag{12}$$

$$C_K = \mathbb{C}(M^{-1})^T, \ B_K = N^{-1}\mathbb{B}.$$

(Proof) CLS

$$\left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right] = \left[\begin{array}{cc|c} A & B_2 C_K & B_1 \\ B_K C_2 & A_K & B_K D_{21} \\ \hline C_1 & D_{12} C_K & 0 \end{array}\right].$$

According to Lemma 1, the $\mathcal{H}_2$ control problem has a solution iff $\exists P = P^T$ and $W = W^T$ satisfying

$$\begin{bmatrix} PA_c + A_c^T P & C_c^T \\ C_c & -I \end{bmatrix} < 0, \quad \begin{bmatrix} W & B_c^T P \\ PB_c & P \end{bmatrix} > 0, \quad \mathrm{Tr}(W) < \gamma^2.$$

Factorize matrix P as

$$P\Pi_1 = \Pi_2, \quad \Pi_1 = \begin{bmatrix} X & I \\ M^T & 0 \end{bmatrix}, \ \Pi_2 = \begin{bmatrix} I & Y \\ 0 & N^T \end{bmatrix}.$$

Congruent transformations lead to equivalent inequalities

$$\begin{bmatrix} \Pi_2^T A_c \Pi_1 + (\Pi_2^T A_c \Pi_1)^T & (C_c \Pi_1)^T \\ C_c \Pi_1 & -I \end{bmatrix} < 0, \ \begin{bmatrix} W & (\Pi_2^T B_c)^T \\ \Pi_2^T B_c & \Pi_2^T \Pi_1 \end{bmatrix} > 0.$$

After concrete computation,

$$\Pi_2^T A_c \Pi_1 = \begin{bmatrix} AX + B\mathbb{C} & A + B\mathbb{D}C \\ \mathbb{A} & YA + \mathbb{B}C \end{bmatrix}, \quad \Pi_2^T B_c = \begin{bmatrix} B_1 \\ YB_1 + \mathbb{B}D_{21} \end{bmatrix}$$

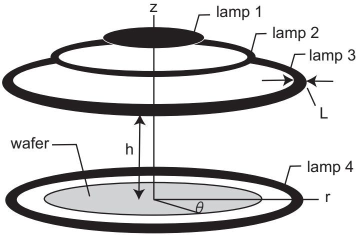
$$C_c \Pi_1 = [C_1 X + D_{12}\mathbb{C} \ \ C_1], \quad \Pi_2^T \Pi_1 = \begin{bmatrix} X & I \\ I & Y \end{bmatrix}$$

and the conclusion is obtained. ∇

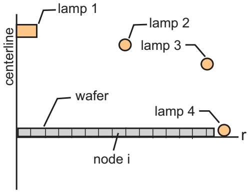
Case Study: $\mathcal{H}_2$ control of a RTP system

1. RTP (rapid thermal processing) system is used for the temperature treatment of semiconductor wafers.
2. Besides control design, locations of actuator and sensor are also vital in achieving high performance.
3. Three circular lamp zones on top of the wafer and one circumvallating the wafer.
4. Lamps 1-3 are for the thermal processing of wafer surface while Lamp 4 is used to compensate for the heat leaking from the side of wafer.
5. Pyrometers are installed below the bottom of wafer to measure the temperature.

Lamp location



(a) 3D illustration



(b) Cross section

Figure: RTP system

Model of RTP

1. Heat conduction equation of wafer

$$\frac{\partial}{r\partial r}\left(kr\frac{\partial T}{\partial r}\right)+\frac{\partial}{r^2\partial\theta}\left(k\frac{\partial T}{\partial\theta}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)=\rho C\frac{\partial T}{\partial t}. \tag{13}$$

2. Finite element model:
divide the wafer as 20 concentric annular zones with equal areas.

$$m_i C_i\frac{dT_i}{dt}=q_i^{\mathrm{ab}}+q_i^{\mathrm{em}}+q_i^{\mathrm{conv}}+q_i^{\mathrm{cond}},\ \ i=1,\cdots,n \tag{14}$$

Model of RTP

1. Emission heat q_i^{em}

$$q_i^{\mathrm{em}} = -\varepsilon_i \sigma A_i T_i^4, \quad \varepsilon_i = \frac{0.7128}{1 + \exp(\frac{T_i - 666.15}{-64.70})} \tag{15}$$

2. Convection heat q_i^{conv}

$$q_i^{\mathrm{conv}} = -h_i A_i (T_i - T_{\mathrm{gas}}), \quad h_i = 14.2 + 8.6 \left(\frac{r_i}{R} \right) \tag{16}$$

3. Conduction heat q_i^{cond}

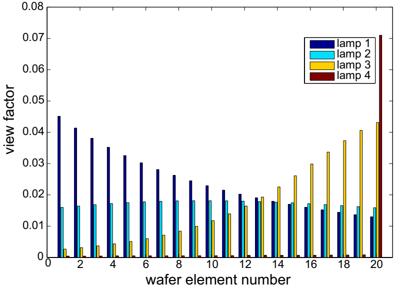
$$q_i^{\mathrm{cond}} = -2\pi k_i Z \left(\frac{T_i - T_{i-1}}{r_i^{\mathrm{cen}} - r_{i-1}^{\mathrm{cen}}} r_i^{\mathrm{in}} + \frac{T_i - T_{i+1}}{r_i^{\mathrm{cen}} - r_{i+1}^{\mathrm{cen}}} r_i^{\mathrm{out}} \right) \tag{17}$$

4. Radiation heat q_i^{ab} of lamp

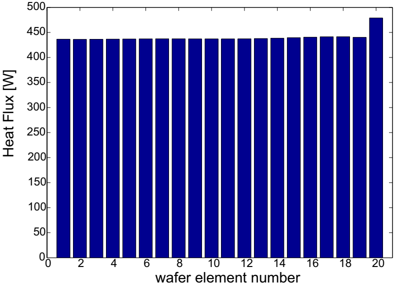
$$q_i^{\mathrm{ab}} = \alpha_i \sum_{j=1}^{J} L_{i,j} P_j. \tag{18}$$

Optimal configuration of lamps

1. To realize a high uniformity in the surface temperature, the heat irradiated from the lamps to the wafer surface must be uniform.
2. Configuration of lamps needs to be optimized.



(a) View factors between annular zones and lamps



(b) Radiation heat on wafer surface

Location of sensors

1. What is important is not only the location of actuator, but also that of the sensor.
2. Only finite points are measured. So, it is important to determine the number of sensors and locate them suitably. In correspondence with the number lamp zones, 4 pyrometers are used.
3. Sensors are located at the annular zones numbered 3, 11, 18 and 20 where the peaks of temperature fluctuation occur.

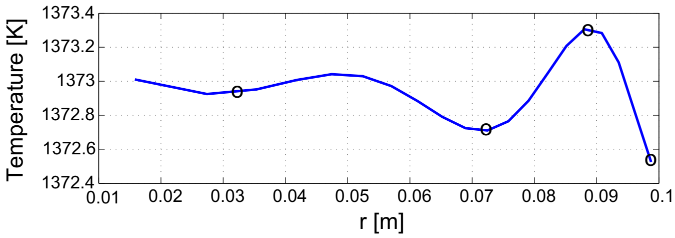


Figure: Steady-state temperature distribution of wafer surface

Specification

1. In the whole thermal processing, the maximal temperature fluctuation must be less than ± 1 K.

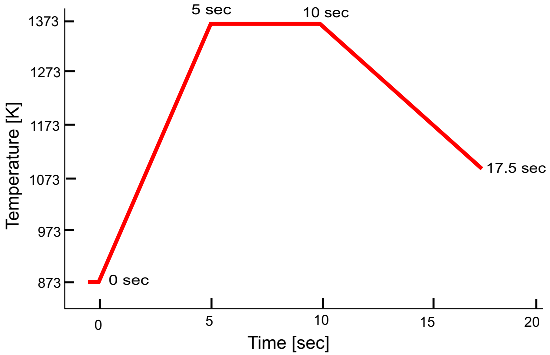


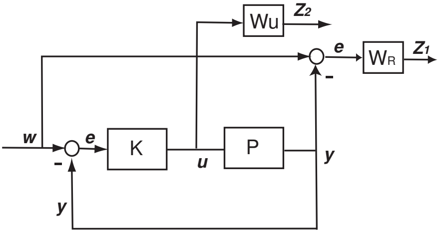
Figure: Reference trajectory of the thermal processing

$\mathcal{H}_2$ control design

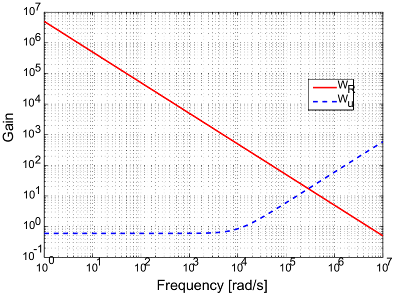
1. $T = [T_1, \cdots, T_{20}]^T$: temperature, $u = [u_1, \cdots, u_4]^T$: lamp power
2. Linear approximation about the target temperature 1373 K

$$\Delta\dot{T} = A\Delta T + B\Delta u, \quad y = C\Delta T \tag{19}$$

3. $\Delta T = T - T_0$, $\Delta u = u - u_0$: error w.r.t. the equilibrium $(T_0,\ u_0)$

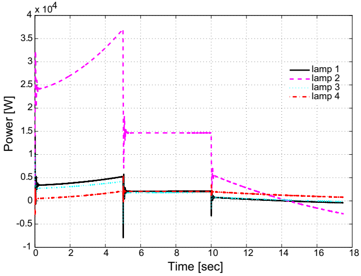


(a) Generalized plant

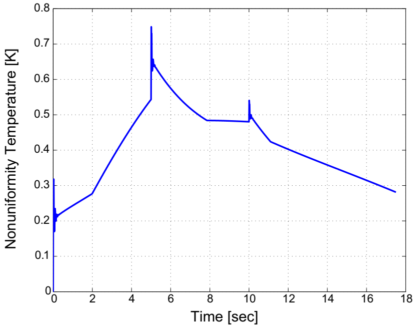


(b) Weighting functions W_R, W_u

Simulation Results: effect of sensor location



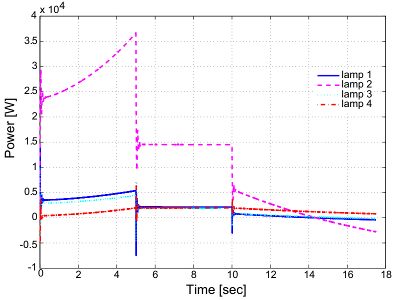
(a) Lamp powers



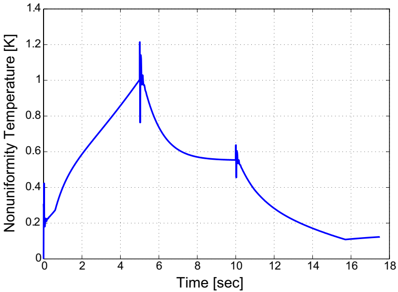
(b) Nonuniformity of temperature

Figure: Result of optimal sensor location

Simulation Results: effect of sensor location



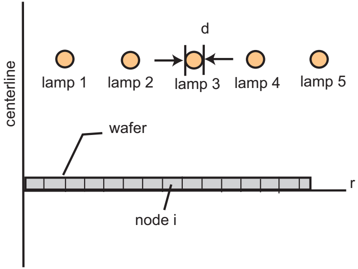
(a) Lamp powers



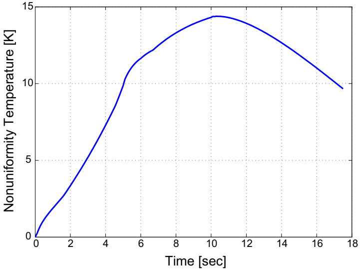
(b) Nonuniformity of temperature

Figure: Result of equal distance sensor location

Simulation Results: effect of lamp location



(a) Lamp configuration



(b) Nonuniformity of temperature

Figure: RTP with lamps located at the same height and equal distance