Chapter 13

Robustness Analysis 2
Lyapunov Method

Table of contents

1. Overview of Lyapunov Stability Theory
   - Asymptotic Stability Condition
   - Condition for State Convergence Rate
2. Quadratic Stability
   - Condition for Quadratic Stability
   - Polytopic Systems
   - Norm-Bounded Parametric Systems
3. Passive Systems
   - Strongly PR case
   - Strictly PR case

Lyapunov Stability Theory

Nonlinear system (state vector $x \in \mathbb{R}^n$)

$$\dot{x} = f(x), \quad x(0) \neq 0. \tag{1}$$

1. How to find a condition to ensure the asymptotic stability?
2. Lyapunov's idea: not to investigate the state trajectory directly, but to examine the variation of energy instead.
3. No external energy is supplied to system (1), so the motion must stop when the internal energy becomes zero.
4. If we know whether the internal energy converges to zero, we can definitely judge if the state converges to the origin or not.

Lyapunov Stability Theory

1. As an energy function, we use a positive definite function called Lyapunov function
$$V(x) > 0 \quad \forall \; x \neq 0. \tag{2}$$
2. If its time derivative satisfies
$$\dot{V}(x) < 0 \quad \forall x \neq 0, \tag{3}$$
then the convergence of state is guaranteed
$$\lim_{t \to \infty} x(t) = 0.$$

Linear case

$$\dot{x} = Ax, \quad x(0) \neq 0. \tag{4}$$

1. Lyapunov function

$$V(x) = x^T P x > 0 \quad \forall x \neq 0. \tag{5}$$

2. Differentiation of $V(x) = x^T P x$ along the trajectory of $\dot{x} = Ax$

$$\begin{aligned} \dot{V}(x) &= \dot{x}^T P x + x^T P \dot{x} = (Ax)^T P x + x^T P (Ax) \\ &= x^T (A^T P + PA) x. \end{aligned} \tag{6}$$

3. So

$$\dot{V}(x) < 0 \;\Leftrightarrow\; A^T P + PA < 0. \tag{7}$$

Theorem 1

Linear system (1) is asymptotically stable iff there exists a $P > 0$ satisfying (7).

Condition for State Convergence Rate

1. How to guarantee a convergence rate of state?
2. When the LMI

$$A^T P + PA + 2\sigma P < 0, \quad \sigma > 0. \tag{8}$$

has a positive definite solution P,

$$\dot{V}(x) = x^T(A^T P + PA)x < x^T(-2\sigma P)x = -2\sigma V(x).$$

3. Solution of $\dot{y} = -2\sigma y$ is $y(t) = e^{-2\sigma t} y(0)$.
4. According to the comparison principle, $V(x)$ satisfies

$$V(x(t)) < e^{-2\sigma t} V(x(0)).$$

5. Since $\lambda_{\min}(P)\left\|x(t)\right\|^2 \leq x^T(x)Px(t) < e^{-2\sigma t}x^T(0)Px(0) \leq e^{-2\sigma t}\lambda_{\max}(P)\left\|x(0)\right\|^2$

$$\left\|x(t)\right\| < \sqrt{\lambda_{\max}(P)/\lambda_{\min}(P)}\left\|x(0)\right\| e^{-\sigma t}, \tag{9}$$

6. $x(t)$ converges to zero at a rate higher than σ.

Quadratic Stability

1. Uncertain system
$$\dot{x} = A(\theta)x, \quad x(0) \neq 0 \tag{10}$$
$\theta \in \mathbb{R}^p$ is a bounded vector of uncertain parameters.
2. Example: mass-spring-damper system $(u = 0)$
$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} x = A(m, b, k)x$$
Parameter vector $\theta = [m \;\; b \;\; k]^T$.
3. Barmish's idea: use a common quadratic function $V = x^T P x$ to investigate the stability for the entire system set
$$V(x) = x^T P x > 0 \;\forall x \neq 0; \quad \dot{V}(x, \theta) < 0 \;\forall x \neq 0, \;\theta. \tag{11}$$
4. When this is possible, the system set is said to be *quadratically stable*.
5. Although a very strong spec, quadratic stability is quite effective in engineering applications.

Condition for Quadratic Stability

1. From $\dot{V}(x,\theta) = x^T(A^T(\theta)P + PA(\theta))x$, quadratic stability condition is $\exists P > 0$ satisfying

$$A^T(\theta)P + PA(\theta) < 0 \ \forall\theta. \tag{12}$$

2. Question: how to calculate a solution P for inequality (12)?
3. No general solution exists. Results known for two classes of $A(\theta)$

Example 1

$$\dot{x} = -(2+\theta)x, \ \ \theta > -2.$$

Since $A^T(\theta)P + PA(\theta) = -(2+\theta)P - P(2+\theta) = -2(2+\theta)P$,

$$A^T(\theta)P + PA(\theta) = -2(2+\theta) < 0 \quad \forall\theta \in (-2, \ \infty)$$

w.r.t. $P = 1$. *Therefore, the stability is guaranteed.*

Polytopic Systems

$$\dot{x} = (\sum_{i=1}^{N} \lambda_i A_i)x, \quad x(0) \neq 0 \tag{13}$$

1. Uncertain parameters satisfy $\lambda_i \geq 0$, $\sum_{i=1}^{N} \lambda_i = 1$.
2. Quadratic stability condition

$$(\sum_{i=1}^{N} \lambda_i A_i)^T P + P(\sum_{i=1}^{N} \lambda_i A_i) < 0 \quad \forall \lambda_i$$

$$\Leftrightarrow \sum_{i=1}^{N} \lambda_i (A_i^T P + PA_i) < 0 \quad \forall \lambda_i. \tag{14}$$

3. This inequality must hold at all vertices of the polytope. Hence,

$$A_i^T P + PA_i < 0 \quad \forall i = 1, \ldots, N \tag{15}$$

$A_i^T P + PA_i < 0$ is the condition for $\lambda_i = 1, \lambda_j = 0$ $(j \neq i)$

Polytopic Systems

1. As all λ_i are nonnegative and their sum is 1, at least one of them must be positive.
2. So when (15) holds, we have

$$\sum_{i=1}^{N} \lambda_i (A_i^T P + P A_i) < 0$$

3. LMI conditions (15) at all vertices are equivalent to the quadratic stability condition (12).

Example: mass-spring-damper system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} x.$$

1. Parameter set

$$1 \leq m \leq 2, \quad 10 \leq k \leq 20, \quad 5 \leq b \leq 10.$$

2. $\theta = [m \;\; b \;\; k]^T$ forms a cube with eight vertices.
3. Quadratic stability condition (15) has a solution

$$P = \begin{bmatrix} 1.9791 & -2.8455 \\ -2.8455 & 14.2391 \end{bmatrix} > 0.$$

4. So the system is quadratically stable.
5. This conclusion is very natural in view of the fact that the damping coefficient b is positive.

Example: mass-spring-damper system

1. On the other hand, when the damping coefficient ranges over $0 \leq b \leq 5$, the solution of (15) becomes

$$P = \begin{bmatrix} 0.85 & 0.9 \\ 0.9 & 10.26 \end{bmatrix} \times 10^{-11} \approx 0$$

   which is not positive definite.

2. So we cannot draw the conclusion that this system is quadratically stable.
3. In fact, this system set includes a case of zero damping. So the system set is not quadratically stable.

A generalization

1. Parameter-dependent Lyapunov function may reduce the conservatism.
2. A simple example:
$$\dot{x} = A(\theta)x = (A_0 + \theta A_1)x, \quad \theta \in [\theta_m,\ \theta_M].$$
3. In view of the structure of $A(\theta)$, we use a matrix
$$P(\theta) = P_0 + \theta P_1.$$
4. Then
$$P(\theta)A(\theta) = P_0A_0 + \theta^2 P_1A_1 + \theta(P_1A_0 + P_0A_1).$$
5. Due to θ^2, the polytopic structure is destroyed s.t. the stability condition cannot be reduced to the vertex conditions. In LMI approach, so far there is no good solution for problems like this.

1. Method of Gahinet et al.:

$$V(x,\theta) = x^T P(\theta) x, \quad P(\theta) > 0.$$

2. Its derivative is a quadratic function of θ:

$$\begin{aligned}\dot{V}(x,\theta) = & x^T[(A_0^T P_0 + P_0 A_0) + \theta^2 (A_1^T P_1 + P_1 A_1) \\ & + \theta(P_1 A_0 + P_0 A_1 + A_0^T P_1 + A_1^T P_0)]x\end{aligned}$$

3. If $\dot{V}(x,\theta)$ is convex in θ, vertex conditions

   $$A(\theta_m)^T P(\theta_m) + P(\theta_m) A(\theta_m) < 0, \; A(\theta_M)^T P(\theta_M) + P(\theta_M) A(\theta_M) < 0$$

   ensures $\dot{V}(x,\theta) < 0$.

4. Condition for convexity

$$\frac{d^2}{d\theta^2} \dot{V}(x,\theta) = 2x^T (A_1^T P_1 + P_1 A_1) x \geq 0 \;\Rightarrow\; A_1^T P_1 + P_1 A_1 \geq 0.$$

5. Lastly, $P(\theta) > 0$ is guaranteed by the vertex conditions

$$P(\theta_m) > 0, \quad P(\theta_M) > 0.$$

Norm-Bounded Parametric Systems

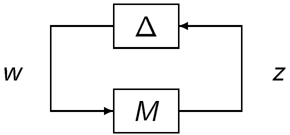
1. Polytopic model is very effective in robustness analysis, but not good for design.
2. Norm-bounded parametric systems

$$M \left\{ \begin{array}{l} \dot{x} = Ax + Bw \\ z = Cx + Dw \end{array} \right. \qquad w = \Delta z, \ \|\Delta(t)\|_2 \leq 1. \tag{16}$$

3. State equation of CLS

$$\dot{x} = (A + B\Delta(I - D\Delta)^{-1} C)x, \quad \|\Delta(t)\|_2 \leq 1. \tag{17}$$

4. When $\Delta(t)$ varies freely in $\|\Delta(t)\|_2 \leq 1$, the invertible condition for $I - D\Delta$ is $\|D\|_2 < 1$ (Exercise 13.2).



Norm-Bounded Parametric Systems

Time-varying version of small-gain theorem (Exercise 13.3) yields that the CLS (M, Δ) is quadratically stable w.r.t. Lyapunov function $V(x) = x^T P x$ if there is $P > 0$ satisfying

$$\begin{bmatrix} A^T P + PA & PB & C^T \\ B^T P & -I & D^T \\ C & D & -I \end{bmatrix} < 0, \tag{18}$$

Theorem 2

The time-varying system (17) is quadratically stable iff there exists a positive definite matrix P satisfying (18).

Example: mass-spring-damper system

$$m = m_0(1 + w_1\delta_1),\ k = k_0(1 + w_2\delta_2),\ b = b_0(1 + w_3\delta_3), \quad |\delta_i| \le 1$$

$$w_1 = \frac{m_{\max}}{m_0} - 1,\ w_2 = \frac{k_{\max}}{k_0} - 1,\ w_3 = \frac{b_{\max}}{b_0} - 1.$$

After normalizing $\Delta = [\delta_1 \quad \delta_2 \quad \delta_3]$, we have

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k_0}{m_0} & -\frac{b_0}{m_0} \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ C = -\sqrt{3} \begin{bmatrix} \frac{k_0}{m_0} w_1 & \frac{b_0}{m_0} w_1 \\ \frac{k_0}{m_0} w_2 & 0 \\ 0 & \frac{b_0}{m_0} w_3 \end{bmatrix}$$

$$D = -\sqrt{3} \begin{bmatrix} w_1 & 0 & 0 \end{bmatrix}^T.$$

1. When $1 \le m \le 2,\ 10 \le k \le 20,\ 5 \le b \le 10$, (18) has a solution
$$P = \begin{bmatrix} 1.9791 & -2.8455 \\ -2.8455 & 14.2391 \end{bmatrix} > 0.$$
2. When $0 \le b \le 5$, no solution exists for (18) and $P > 0$.

Proof

Sufficiency:

$$\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x} = (Ax + Bw)^T P x + x^T P(Ax + Bw)$$
$$= \begin{bmatrix} x \\ w \end{bmatrix}^T \begin{bmatrix} A^T P + PA & PB \\ B^T P & 0 \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix}. \qquad (19)$$

$\|\Delta(t)\|_2 \leq 1$ implies $w^T w = z^T \Delta^T \Delta z \leq z^T z$. As $z = Cx + Dw$, we get

$$U(x,w) = \begin{bmatrix} x \\ w \end{bmatrix}^T \left\{ \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} C^T \\ D^T \end{bmatrix} [C \quad D] \right\} \begin{bmatrix} x \\ w \end{bmatrix} \leq 0. \qquad (20)$$

It can be proved that $x \neq 0$ in any nonzero vector $\begin{bmatrix} x \\ w \end{bmatrix}$ satisfying $U(x,w) \leq 0$.

(18) is equivalent to (Schur's lemma)

$$\begin{aligned} 0 &> \begin{bmatrix} A^T P + PA & PB \\ B^T P & -I \end{bmatrix} + \begin{bmatrix} C^T \\ D^T \end{bmatrix} [C \quad D] \\ &= \begin{bmatrix} A^T P + PA & PB \\ B^T P & 0 \end{bmatrix} - \left\{ \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} C^T \\ D^T \end{bmatrix} [C \quad D] \right\}. \end{aligned}$$

Multiplying this inequality by $\begin{bmatrix} x \\ w \end{bmatrix} \neq 0$, we have

$$\dot{V}(x) < U(x, w) \leq 0.$$

So the quadratic stability is proved.

Necessity: when the system is quadratically stable,

$$\dot{V}(x) < 0, \quad U(x, w) \leq 0$$

hold simultaneously for $x \neq 0$. For a bounded $\begin{bmatrix} x \\ w \end{bmatrix}$, $\dot{V}(x)$ and $U(x, w)$ are also bounded. Enlarging $\dot{V}(x)$ suitably by a factor $\rho > 0$, we have

$$\rho \dot{V}(x) < U(x, w) \quad \forall x \neq 0.$$

Finally, absorbing ρ into P and renaming ρP as P, we obtain

$$\dot{V}(x) - U(x, w) < 0 \quad \forall \begin{bmatrix} x \\ w \end{bmatrix} \neq 0.$$

This inequality is equivalent to (18). ∇

Passive Systems

1. A system is called *passive* if its transfer function is either PR, or strongly PR, or strictly PR.
2. CLS: uncertainty $\Delta(s)$ is PR while the nominal CLS $M(s)$ is either strongly PR or strictly PR.
3. Intuitively, the phase angle of a PR system is limited to $[-90^\circ, 90^\circ]$ and that of a strongly PR system restricted to $(-90^\circ, 90^\circ)$. So the phase angle of the open-loop system is always not $\pm 180^\circ$ and the stability of CLS may be expected.

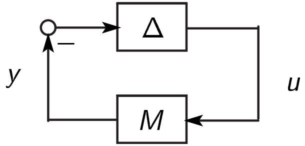


Figure: Closed-loop system with a PR uncertainty

Passive Systems

Theorem 3

Assume that the uncertainty $\Delta(s)$ is stable and PR. Then, the CLS is asymptotically stable if the nominal system $M(s)$ is stable and strongly PR.

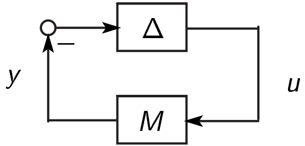


Figure: Closed-loop system with a PR uncertainty

(Proof) Let the state equations of M and Δ be

$$\begin{aligned}\Delta(s):\quad & \dot{x}_1 = A_1x_1 + B_1(-y),\ u = C_1x_1 + D_1(-y)\\ M(s):\quad & \dot{x}_2 = A_2x_2 + B_2u,\ y = C_2x_2 + D_2u.\end{aligned}$$

According to PR lemma and strongly PR lemma, $\exists\ P > 0, Q > 0$ satisfying

$$\begin{bmatrix} A_1^TP + PA_1 & PB_1 \\ B_1^TP & 0 \end{bmatrix} - \begin{bmatrix} 0 & C_1^T \\ C_1 & D_1 + D_1^T \end{bmatrix} \le 0 \tag{21}$$

$$\begin{bmatrix} A_2^TQ + QA_2 & QB_2 \\ B_2^TQ & 0 \end{bmatrix} - \begin{bmatrix} 0 & C_2^T \\ C_2 & D_2 + D_2^T \end{bmatrix} < 0 \tag{22}$$

Then, for $V_1(x_1) = x_1^TPx_1 > 0,\ V_2(x_2) = x_2^TQx_2 > 0$ we have

$$\dot{V}_1(x_1) \le -u^Ty - y^Tu, \qquad \dot{V}_2(x_2) < u^Ty + y^Tu.$$

Lyapunov candidate of CLS: $V(x_1, x_2) = V_1(x_1) + V_2(x_2)$

$$\dot{V}(x_1, x_2) = \dot{V}_1(x_1) + \dot{V}_2(x_2) < 0$$

Therefore, the CLS is asymptotically stable.

Passive Systems

Theorem 4

Assume that the uncertainty $\Delta(s)$ is stable and PR. The CLS is asymptotically stable if the nominal system $M(s)$ is stable and there is a constant $\epsilon > 0$ such that $M(s - \epsilon)$ is PR.

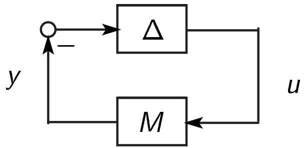


Figure: Closed-loop system with a PR uncertainty

(Proof) The proof is similar to that of Theorem 3. The only difference is to replace the strongly PRness of $M(s)$ by (modified) strictly PRness, i.e.

$$\begin{bmatrix} (A_2+\epsilon I)^T Q + Q(A_2+\epsilon I) & QB_2 \\ B_2^T Q & 0 \end{bmatrix} - \begin{bmatrix} 0 & C_2^T \\ C_2 & 0 \end{bmatrix} \leq 0. \tag{23}$$

$$\dot{V}_2(x_2) \leq u^T y + y^T u - 2\epsilon x_2^T Q x_2 \tag{24}$$

So again, the Lyapunov candidate $V(x_1, x_2) = V_1(x_1) + V_2(x_2)$ satisfies

$$\dot{V}(x_1, x_2) = \dot{V}_1(x_1) + \dot{V}_2(x_2) \leq -2\epsilon x_2^T Q x_2$$

When x_2 is not identically zero, $V(x_1, x_2)$ strictly decreases.
When $x_2(t) \equiv 0$, $y = C_2 x_2 = 0$. Substituting $y = 0$ into $\dot{x}_1$, we have

$$\dot{x}_1 = A_1 x_1 \Rightarrow x_1(t) \rightarrow 0$$

because A_1 is stable. Therefore, the CLS is asymptotically stable.