Chapter 3

Basics of Linear Matrix Inequality

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LMI

- LMI
$$F(x) = F_0 + \sum_{i=1}^{m} x_i F_i > 0 \quad (1)$$
$x \in \mathbb{R}^m$: unknown vector, $F_i = F_i^T \in \mathbb{R}^{n\times n}$: constant matrix.
- $F(x)$ is positive definite, i.e., $u^T F(x) u > 0$ for all non-zero vector u.
- LMI can be solved numerically by the interior point method.
- MATLAB has an LMI toolbox tailored for solving control problems.

Control Problem and LMI

Example 1

2D system $\dot{x}(t) = Ax(t)$ *is stable iff* $\exists P = P^T > 0$ *satisfying*

$$AP + PA^T < 0.$$

Description in terms of matrix basis

$$P = \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix} = x_1 P_1 + x_2 P_2 + x_3 P_3$$

$$P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ P_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ P_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow x_1(AP_1 + P_1A^T) + x_2(AP_2 + P_2A^T) + x_3(AP_3 + P_3A^T) < 0.$$

This is an LMI about $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}$.

Typical LMI Problems

- Feasibility problem: LMIP
  Seek a vector x^* for LMI $F(x^*) < 0$
- Eigenvalue problem: EVP
  1. Optimization subject to LMI constraints

$$\begin{aligned} &\text{minimize } \lambda \\ &\text{subject to } \lambda I - A(x) > 0, \ B(x) > 0. \end{aligned} \tag{2}$$

  2. $A(x), B(x)$: symmetric and affine matrix functions of vector x.
  3. Minimizing λ subject to $\lambda I - A(x) > 0$ can be interpreted as minimizing the largest eigenvalue of matrix $A(x)$ because the largest λ that does not meet $\lambda I - A(x) > 0$ is the maximal λ satisfying $\det(\lambda I - A(x)) = 0$.

Typical LMI Problems

- Generalized eigenvalue problem: GEVP

$$\begin{aligned} &\text{minimize } \lambda \\ &\text{subject to } \lambda B(x) - A(x) > 0, \;\; B(x) > 0, \;\; C(x) > 0 \end{aligned} \tag{3}$$

1. $A(x), B(x), C(x)$ are symmetric and affine matrix functions of vector x.
2. Equivalent to minimizing the largest generalized eigenvalue satisfying $\det(\lambda B(x) - A(x)) = 0$.

An example

State feedback stabilization problem:

$$\dot{x} = Ax + Bu, \quad u = Fx$$

1. Closed-loop system
$$\dot{x} = (A + BF)x.$$
2. Stability condition: there exist matrices $P > 0$ and F satisfying
$$(A + BF)P + P(A + BF)^T < 0.$$
3. Product FP appears, called *bilinear matrix inequality* (BMI).
4. BMI problem is non-convex and very difficult to solve numerically.
5. Viewpoint of system control: whether a system can be stabilized depends on the controllability of (A, B), not the control gain F. There must be a stabilization condition independent of F.

From BMI to LMI: Variable Elimination

Theorem 1

Given real matrices E, F, G with G being symmetric. There is a matrix X satisfying the inequality

$$E^T XF + F^T X^T E + G < 0 \tag{4}$$

iff the following two inequalities hold simultaneously

$$E_\perp^T GE_\perp < 0, \quad F_\perp^T GF_\perp < 0. \tag{5}$$

Stabilization example: continued

1. Stability condition:

$$(AP + PA^T) + PF^TB^T + BFP < 0, \quad P > 0.$$

2. For this inequality to have a solution F,

$$(B^T)_\perp^T(AP + PA^T)(B^T)_\perp < 0$$

must hold. This condition only depends on matrix P, so it is an LMI.

3. Since $P_\perp$ does not exist, we do not need to consider the second inequality of Theorem 1.

4. After obtaining P, we may substitute it back into the first inequality. Then, the inequality becomes an LMI about F and can be solved numerically.

A matrix construction problem

Lemma 1

Given two positive definite matrices $X, Y \in \mathbb{R}^{n\times n}$ and a positive integer r, there is a positive definite matrix $P \in \mathbb{R}^{(n+r)\times(n+r)}$ satisfying

$$P = \begin{bmatrix} Y & * \\ * & * \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} X & * \\ * & * \end{bmatrix}$$

iff

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0, \ \mathrm{rank} \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \leq n + r.$$

Further, a solution is given by

$$P = \begin{bmatrix} Y & F \\ F^T & I \end{bmatrix}, \quad FF^T = Y - X^{-1}, \ F \in \mathbb{R}^{n\times r}.$$

Example: output feedback stabilization

1. Plant
$$\dot{x}_P = Ax_P + Bu \tag{6}$$
$$y = Cx_P + Du \tag{7}$$
2. Dynamic output feedback controller
$$\dot{x}_K = A_K x_K + B_K y \tag{8}$$
$$u = C_K x_K + D_K y. \tag{9}$$
3. Closed-loop system
$$\begin{bmatrix} \dot{x}_P \\ \dot{x}_K \end{bmatrix} = A_c \begin{bmatrix} x_P \\ x_K \end{bmatrix}, \quad A_c = \begin{bmatrix} A + BD_K C & BC_K \\ B_K C & A_K \end{bmatrix}. \tag{10}$$
4. Stability condition: $\exists \mathbb{P}$ satisfying inequalities
$$A_c^T \mathbb{P} + \mathbb{P} A_c < 0, \quad \mathbb{P} > 0. \tag{11}$$
5. Problem: find the solvability condition.

Solution

- Collect all controller coefficient matrices into a single matrix

$$A_c = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} D_K & C_K \\ B_K & A_K \end{bmatrix} \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix} = \overline{A} + \overline{B}\mathcal{K}\overline{C}$$

$$\mathcal{K} = \begin{bmatrix} D_K & C_K \\ B_K & A_K \end{bmatrix}, \ \overline{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \ \overline{B} = \begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix}, \ \overline{C} = \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}$$

- Transformed stability condition

$$\mathbb{P}\overline{A} + \overline{A}^T\mathbb{P} + \mathbb{P}\overline{B}\mathcal{K}\overline{C} + (\mathbb{P}\overline{B}\mathcal{K}\overline{C})^T < 0.$$

- Orthogonal matrices

$$\overline{C}_\perp = \begin{bmatrix} C_\perp \\ 0 \end{bmatrix}, \quad (\overline{B}^T\mathbb{P})_\perp = \mathbb{P}^{-1} \begin{bmatrix} (B^T)_\perp \\ 0 \end{bmatrix}.$$

- Equivalent condition (Theorem 1)

$$(\overline{C}_\perp)^T(\mathbb{P}\overline{A} + \overline{A}^T\mathbb{P})\overline{C}_\perp < 0, \quad (\overline{B}^T)_\perp^T(\overline{A}\mathbb{P}^{-1} + \mathbb{P}^{-1}\overline{A}^T)(\overline{B}^T)_\perp < 0.$$

Solution

- As in Lemma 1, we set

$$\mathbb{P} = \begin{bmatrix} Y & * \\ * & * \end{bmatrix}, \quad \mathbb{P}^{-1} = \begin{bmatrix} X & * \\ * & * \end{bmatrix}.$$

- Equivalent condition: $\exists\ X, Y$ meeting LMIs

$$(B^T)_\perp^T(AX + XA^T)(B^T)_\perp < 0 \tag{12}$$

$$(C_\perp)^T(YA + A^TY)C_\perp < 0. \tag{13}$$

- Finally, according to Lemma 1 $\mathbb{P} > 0$ iff matrices $X,\ Y$ satisfy

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0, \ \mathrm{rank} \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \leq n + r. \tag{14}$$

From BMI to LMI: Variable Change

- Variable elimination method handles only one matrix inequality
- More inequalities appear in multi-objective problems
- New method needed

1. Recall the stabilization problem: find matrices $P > 0$ and F satisfying

$$(A + BF)P + P(A + BF)^T < 0.$$

2. Variable change: set FP as a new variable M

$$A^T P + PA + BM + M^T B^T < 0.$$

3. Due to $P > 0$, once we obtain (P, M), the unique solution of F can be calculated from $F = MP^{-1}$.

Variable change: output feedback case

1. Closed-loop system

$$\begin{bmatrix} \dot{x}_P \\ \dot{x}_K \end{bmatrix} = A_c \begin{bmatrix} x_P \\ x_K \end{bmatrix}, \quad A_c = \begin{bmatrix} A + BD_K C & BC_K \\ B_K C & A_K \end{bmatrix}. \tag{15}$$

2. Stability condition

$$A_c^T \mathbb{P} + \mathbb{P} A_c < 0, \quad \mathbb{P} > 0. \tag{16}$$

3. Question: how to transform it to an LMI problem via a suitable variable change?

Structure of $\mathbb{P}$

1. Partition

$$\mathbb{P} = \begin{bmatrix} Y & N \\ N^T & * \end{bmatrix}, \quad \mathbb{P}^{-1} = \begin{bmatrix} X & M \\ M^T & * \end{bmatrix}. \tag{17}$$

2. Since $\mathbb{P}\mathbb{P}^{-1} = I$, there hold

$$\mathbb{P}\begin{bmatrix} X \\ M^T \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix}, \quad \mathbb{P}\begin{bmatrix} I \\ 0 \end{bmatrix} = \begin{bmatrix} Y \\ N^T \end{bmatrix}.$$

3. $\mathbb{P}$ satisfies

$$\mathbb{P}\Pi_1 = \Pi_2, \quad \Pi_1 = \begin{bmatrix} X & I \\ M^T & 0 \end{bmatrix}, \Pi_2 = \begin{bmatrix} I & Y \\ 0 & N^T \end{bmatrix} \tag{18}$$

4. In other words, both $\mathbb{P}$ and $\mathbb{P}^{-1}$ can be described as the quotients of triangular matrices using four matrices (X, Y, M, N).

Variable change

1. Advantage: equivalent inequality

$$\Pi_1^T A_c^T \mathbb{P} \Pi_1 + \Pi_1^T \mathbb{P} A_c \Pi_1 < 0$$

obtained by multiplying (16) with Π_1^T and Π_1 from the left and right.

2. Detailed calculation

$$\begin{aligned}
\Pi_1^T \mathbb{P} A_c \Pi_1 &= \Pi_2^T A_c \Pi_1 \\
&= \begin{bmatrix} I & Y \\ 0 & N^T \end{bmatrix}^T \begin{bmatrix} A + BD_K C & BC_K \\ B_K C & A_K \end{bmatrix} \begin{bmatrix} X & I \\ M^T & 0 \end{bmatrix} \\
&= \begin{bmatrix} AX + B\mathbb{C} & A + B\mathbb{D}C \\ \mathbb{A} & YA + \mathbb{B}C \end{bmatrix}
\end{aligned} \tag{19}$$

3. New unknown matrices $\mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{D}$

$$\begin{aligned}
&\mathbb{A} = NA_K M^T + NB_K CX + YBC_K M^T + Y(A + BD_K C)X \\
&\mathbb{B} = NB_K + YBD_K, \quad \mathbb{C} = C_K M^T + D_K CX, \quad \mathbb{D} = D_K.
\end{aligned} \tag{20}$$

Variable change

- Final stability condition: an LMI about $\mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{D}$:

$$\begin{bmatrix} AX + XA^T + B\mathbb{C} + \mathbb{C}^T B^T & A + B\mathbb{D}C + \mathbb{A}^T \\ A^T + C^T\mathbb{D}^T B^T + \mathbb{A} & YA + A^T Y + \mathbb{B}C + C^T\mathbb{B}^T \end{bmatrix} < 0.$$

- Remaining question: is the variable change one-to-one?
- Controller parameters (when M, N have full row ranks)

$$D_K = \mathbb{D}, \;\; C_K = (\mathbb{C} - D_K CX)(M^\dagger)^T, \;\; B_K = N^\dagger(\mathbb{B} - YBD_K) \quad (21)$$

$$A_K = N^\dagger(\mathbb{A} - NB_K CX - YBC_K M^T - Y(A + BD_K C)X)(M^\dagger)^T.$$

- Strengthened condition for $\mathbb{P} > 0$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0 \;\Rightarrow\; X - Y^{-1} > 0 \;\Rightarrow\; I - XY \text{nonsingular} \quad (22)$$

- there are nonsingular matrices M, N satisfying

$$MN^T = I - XY. \quad (23)$$

This equation comes from the (1,1) block of $\mathbb{P}^{-1}\mathbb{P} = I$.