Chapter 12

Robustness Analysis 1
Small Gain Principle

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Small Gain Theorem

Theorem 1 (Small Gain Theorem)

Assume that $M(s), \Delta(s)$ are stable. The closed-loop system is robustly stable iff one of the following conditions is true.

- *When $\|\Delta\|_\infty \leq 1$, there holds $\|M(s)\|_\infty < 1$.*
- *When $\|\Delta\|_\infty < 1$, there holds $\|M(s)\|_\infty \leq 1$.*

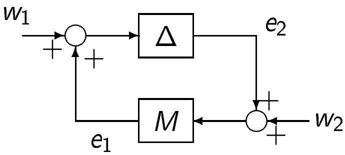


Figure: Small Gain Theorem

Small Gain Theorem: proof

Sufficiency:
under the given condition, the loop gain satisfies

$$|M(j\omega)\Delta(j\omega)| = |M(j\omega)||\Delta(j\omega)| < 1.$$

It is contained inside the unit disk and does not encircle $(-1, j0)$. So, CLS is stable.
Necessity:

- Basic idea: find an uncertainty in the given set which destabilizes CLS if $\|M\|_\infty \geq 1$.
- Due to $\|M\|_\infty \geq 1$ and the continuity of frequency response, there must be a frequency $\omega_0 \in [0, \infty]$ at which $|M(j\omega_0)| = 1$ holds, i.e.

$$M(j\omega_0) = e^{j\theta} \text{ or } -e^{j\theta},\ \theta \in [0, \pi).$$

- In the sequel, Δ is constructed only for the case of positive sign.

Small Gain Theorem: proof

- If we can find an uncertainty $\Delta(s)$ satisfying

$$\Delta(j\omega_0) = e^{-j\theta}, \ \|\Delta\|_\infty \leq 1,$$

- Then

$$1 - M(j\omega_0)\Delta(j\omega_0) = 1 - e^{j\theta}e^{-j\theta} = 0$$

  holds and CLS has an unstable pole $j\omega_0$.
- Next, we construct stable rational uncertainties satisfying this condition case by case.
  1. $M(j\omega_0) = 1$ when $\theta = 0$. Then, $\Delta(s) = 1$.
  2. When $\theta \in (0, \pi)$,

$$\Delta(s) = \frac{a - s}{a + s}, \quad a = \frac{\omega_0}{\tan\theta/2} > 0$$

     satisfies $\Delta(j\omega_0) = e^{-j\theta}$ and $\|\Delta\|_\infty = 1$ simultaneously.
  3. Uncertainties constructed are all stable and rational, and they belong to the given uncertainty set since their $\mathcal{H}_\infty$ norms are 1.

Remarks

1. Physically, small gain theorem corresponds to the fact that the external input is attenuated every time it circulates in the closed loop.
2. Small gain theorem is not only sufficient, but also necessary. That is, for a norm-bounded dynamic uncertainty set, small gain theorem is not conservative.
3. However, we should understand this necessity correctly. It is true only when the phase of uncertainty can change freely, which seldom happens in practice.
4. For real-world systems small gain theorem is simply sufficient, not necessary!

Robust Stability Criteria

- Sensitivity S and Complementary sensitivity T:

$$S(s) = (I + PK)^{-1}, \quad T(s) = (I + PK)^{-1}PK. \tag{1}$$

Table: Robust Stability Criteria

$W(s)$ and $\Delta(s)$ are stable, $\|\Delta\|_\infty \leq 1$	
Plant set $\mathbb{P}$	Robust stability criterion
$(I + \Delta W)P$	Nominal stability and $\|WT\|_\infty < 1$
$(I + \Delta W)^{-1}P$	Nominal stability and $\|WS\|_\infty < 1$
$P + \Delta W$	Nominal stability and $\|WKS\|_\infty < 1$
$P(I + \Delta WP)^{-1}$	Nominal stability and $\|WSP\|_\infty < 1$

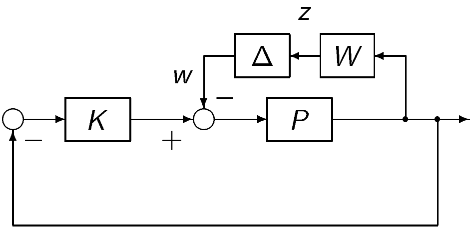
Proof: feedback uncertainty case

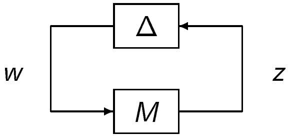
1. When $\Delta(s) = 0$, the system must be nominally stable.
2. Denote the input and output of Δ as z, w resp.
3. Compute the transfer matrix from w to z

$$z = Mw, \quad M = -WSP.$$

4. Transform CLS to the left figure.
5. By small gain theorem the robust stability condition is

$$1 > \|M\|_\infty = \|-WSP\|_\infty = \|WSP\|_\infty .$$

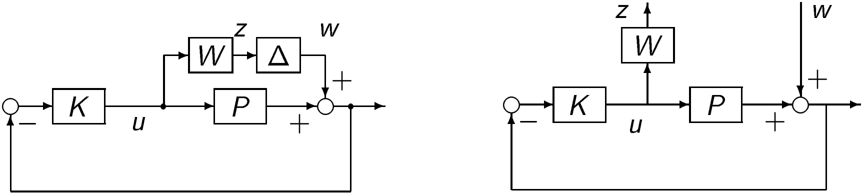




Bridge over performance and robust stability

Example 1: Robust stability about additive uncertainty (Figure (a), $\|\Delta\|_\infty \leq 1$).

$$\text{Robust stability} \;\Leftrightarrow\; (P, K) \text{ is internal stable and } \|WKS\|_\infty < 1$$
$$\Leftrightarrow \text{Suppress the influence of disturbance } w \text{ on input } u \text{ (Figure(b))}$$



(a) Robust stability problem

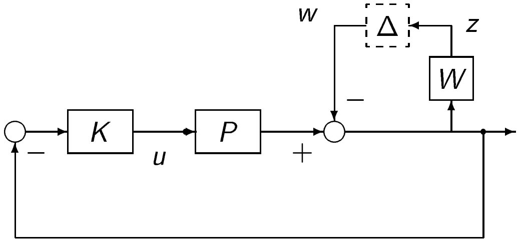
(b) Equivalent disturbance attenuation problem

Bridge over performance and robust stability

Example 2: Sensitivity reduction:

$$(P, K) \text{ is stable and } \|WS\|_\infty < 1 \;\Leftrightarrow$$

$$\text{Robustly stabilize plant set } \{\tilde{P} = \frac{P}{1 + \Delta W},\; \|\Delta\|_\infty \le 1\}$$



- An $\mathcal{H}_\infty$ performance problem is equivalent to a robust stability problem with a virtual uncertainty Δ inserted between the input and output of the closed-loop transfer function.

Bridge over performance and robust stability

Theorem 2

(1) *The CLS containing stable uncertainty* $\|\Delta\|_\infty \leq 1$ *is stable iff the nominal CLS is stable and* $\|\mathcal{F}_\ell(G,K)\|_\infty < 1$.

(2) *The nominal CLS is stable and* $\|\mathcal{F}_\ell(G,K)\|_\infty < 1$ *iff the CLS formed by inserting an arbitrary virtual stable uncertainty* $\|\Delta\|_\infty \leq 1$ *between its input* w *and output* w *is stable.*

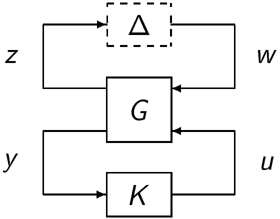


Figure: Equivalence between nominal performance and robust stability

Analysis of Robust Performance

Example 1:

1. plant set

$$\tilde{P} = P + \Delta W, \quad \|\Delta\|_\infty \leq 1.$$

2. Spec: reduce the tracking error

$$\left\| W_S \frac{1}{1 + (P + \Delta W)K} \right\|_\infty < 1. \tag{2}$$

3. Nominal system must satisfy this spec first

$$\|W_S S\|_\infty < 1.$$

4. Robust stability

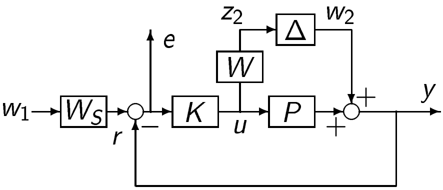
$$\|WKS\|_\infty < 1.$$

Analysis of Robust Performance

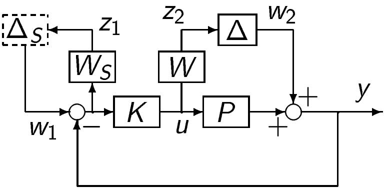
1. Examine if nominal performance and robust stability can guarantee the robust performance.

$$W_S \frac{1}{1+PK} \times \frac{1}{1+\Delta WKS} = W_S S(1+\Delta WKS)^{-1}.$$

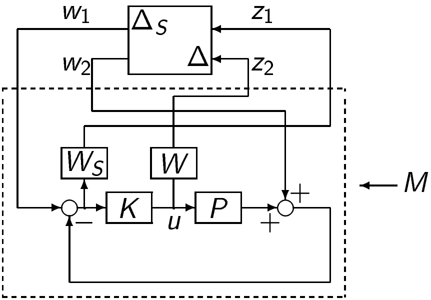
2. Δ ($\|\Delta\|_\infty \leq 1$) can take any complex value. So even if $\|WKS\|_\infty < 1$, such a frequency still can be found at which $|1+\Delta WKS| \ll 1$ holds.
3. For this uncertainty Δ, the tracking performance deteriorates significantly.
4. No matter how good the nominal performance ($\|W_S S\|_\infty$) and robust stability are, the robust performance cannot be ensured!



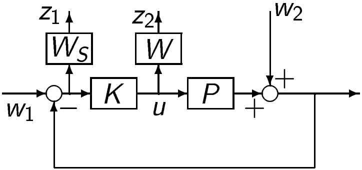
(a) Original problem



(b) Equivalent stability problem



(c) Separation of uncertainty



(d) Conversion to disturbance control problem

Sufficient Condition for Robust Performance

Example 2: Derive a condition for the robust tracking.

1. Robust performance problem is equivalent to the robust stability problem when a virtual uncertainty Δ_S ($\|\Delta_S\|_\infty \leq 1$) (Figure (b))
2. After transforming Figure (b) into Figure (c), the problem is reduced to the robust stability problem of a CLS with a dilated uncertainty:
$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} \Delta_S & \\ & \Delta \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}.$$
3. Since $\left\| \begin{bmatrix} \Delta_S & \\ & \Delta \end{bmatrix} \right\|_\infty < 1$, by small gain theorem a sufficient condition for robust stability is that, transfer matrix $M(s)$ from $[w_1 \;\; w_2]^T$ to $[z_1 \;\; z_2]^T$satisfies
$$\|M\|_\infty \leq 1, \quad M(s) = \begin{bmatrix} W_S S & -W_S S \\ WKS & -WKS \end{bmatrix}.$$
4. This is just a sufficient condition.

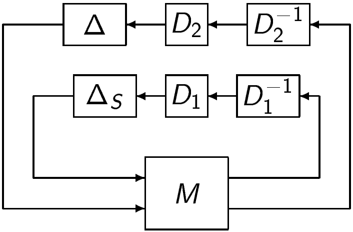
Introduction of Scaling

Example 2: continued

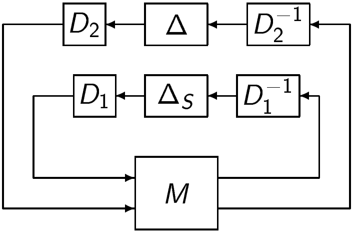
1. Introduction of minimum phase transfer function does not change the stability of CLS.
2. Equivalent block diagram transformations: (a)→(b)→(c)
3. Scaled norm condition for robust performance

$$\left\|D^{-1}MD\right\|_\infty \leq 1, \quad D(s) = \begin{bmatrix} D_1(s) & \\ & D_2(s) \end{bmatrix}. \tag{3}$$

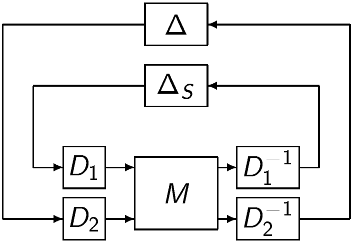
4. A suitable D may make $\left\|D^{-1}MD\right\|_\infty$ less than $\|M\|_\infty$, thus lower the conservatism.



(a) Phase one



(b) Phase two



(c) Phase three

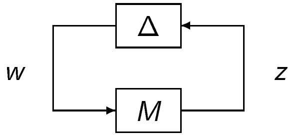
Stability Radius of Norm-Bounded Parametric Systems

Theorem 3 (Qiu's Theorem)

The uncertain system with real uncertainty $\Delta \in \mathbb{R}^{p\times q}$ is robustly stable iff the parameter uncertainty matrix Δ satisfies

$$\frac{1}{\|\Delta\|_2} > \sup_{\omega} \inf_{\gamma\in(0,1]} \sigma_2\left(\begin{bmatrix} \Re(M(j\omega)) & -\gamma\Im(M(j\omega)) \\ \frac{1}{\gamma}\Im(M(j\omega)) & \Re(M(j\omega)) \end{bmatrix}\right). \tag{4}$$

$\sigma_2(X)$ denotes the second largest singular value of matrix X.



Stability radius problem of parametric systems